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НАЦІОНАЛЬНА МЕТАЛУРГІЙНА АКАДЕМІЯ УКРАЇНИ

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HIGHER MATHEMATICS

Part 1

INTRODUCTION TO LINEAR ALGEBRA

**Затверджено на засіданні Вченої ради академії
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Містить курс лекцій англійською мовою з дисципліни “Вища математика” (розділ “Лінійна алгебра”).

Викладені теоретичні основи елементарної лінійної алгебри: теорія матриць та визначників, системи лінійних рівнянь, вектори. Теоретичні положення супроводжуються необхідними поясненнями та ілюстраціями, а також розв’язанням відповідних прикладів.

Призначений для студентів технічних спеціальностей.

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A lectures course in English on discipline “ Higher mathematics” (the section “Linear algebra”) is contained.

The theoretical bases of elementary linear algebra, namely, matrix and determinant theory, systems of linear equations, vectors, are presented. The theoretical principles are furnished with necessary explanations and relevant illustrations, and also with solving of corresponding examples.

Manual is intended for Engineering students.

Fig. 29. Bibliogr.: 7 ref.

Відповідальний за випуск А.В. Павленко, д-р фіз.-мат. наук, проф.

Друкується за авторською редакцією

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INTRODUCTION

Linear algebra is the branch of mathematics concerned with the study of systems of linear equations, vectors, vector (or *linear*) spaces and linear maps (or *linear transformations*). It is an old subject and originally its development dealt with transformation of geometric objects and solution of systems of linear equations. The history of modern linear algebra began in nineteenth century, when W. Hamilton (he is also the author of the term *vector*) in 1843 discovered the *quaternions*, J. Sylvester in 1848 introduced the term *matrix*, and A. Cayley in 1857 developed the matrix theory, one of the most fundamental linear algebraic ideas. In recent years linear algebra has begun to rival calculus as a most commonly used subject in mathematics. It is widely used in different branches of mathematics, in particular, abstract algebra and functional analysis, and also has a concrete representation in analytical geometry and it is generalized in operator theory. Theory and methods of modern linear algebra has also extensive applications to mechanics and engineering, computer science and coding theory, biology and medicine, economics and statistics, and, increasingly, to management and social sciences. The general method of finding a linear approach to the problem, expressing in the terms of linear algebra, and solving it, is one of the most widely used, because nonlinear models can often be approximated by a linear one, and the leaving from nonlinear problems is very important for practice.

This textbook is a basic introduction to the principal ideas and techniques of linear algebra and is intended for students of technical specialization. The first part of this text is dedicated to matrices, determinants and solving of linear systems. In spite of the fact that historically the early emphasis was on the determinant, not the matrix, in modern treatments of linear algebra matrices are considered first. We acted in the same way. The second part is an introduction to vectors and include the basic concepts of vector algebra and its some applications to problems of geometry and physics.

C O N T E N T S

Section 1
MATRICES, DETERMINANTS AND SYSTEMS OF LINEAR ALGEBRAIC EQUATIONS

Section 2
VECTORS

Section 1

MATRICES, DETERMINANTS AND SYSTEMS OF LINEAR ALGEBRAIC EQUATIONS

1. Matrices

Historical reference. The study of matrices is quite old. Latin squares and magic squares have been studied since prehistoric times.

Matrices have a long history of application in solving linear equations. An important Chinese text from between 300 BC and AD 200, Nine Chapters of the Mathematical Art (Chiu Chang Suan Shu), is the first example of the use of matrix methods to solve simultaneous equations. The term “matrix” was first coined in 1848 by J.J. Sylvester. Cayley, Hamilton, Grasmann, Frobenius and von Neumann are among the famous mathematicians who have worked on matrix theory.

1.1. Fundamental concepts

A ***matrix A*** is a rectangular table of real or complex numbers or, more generally, a table consisting of abstract quantities (for example, vectors, functions) that can be added and multiplied. The horizontal lines in a matrix are called ***rows*** and the vertical lines are called ***columns.*** A matrix with m rows and n columns is called an m-by-n matrix (written $m \times n$) or a matrix of size $m \times n$. Numbers n and m are called ***dimensions*** of matrix A. They are always given with the number of rows first, then the number of columns.

A matrix is usually written in the form

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \qquad (1.1)$$

The entry of a matrix A that lies in the i-th row and j-th column is called the i, j entry or (i, j)-th entry of A. This is written as a_{ij}. The convention is that the first index denotes the row and the second index denotes the column. Therefore matrix A is often written simply in the form (a_{ij}). Entries a_{ij} are usually called the ***elements*** or ***components*** of matrix A. The elements $a_{i1}, a_{i2}, \ldots, a_{in}$ are the elements of the i-th row of A, and the elements $a_{1j}, a_{2j}, \ldots, a_{mj}$ are the elements of the j-th column.

A matrix where one of the dimensions equals one is often called a ***vector***. A ***row vector*** or ***row matrix*** is a $1 \times n$ matrix (one row and n columns) $(a_{11} \quad a_{12} \quad \ldots \quad a_{1n})$ while a ***column vector*** or ***column matrix*** is an $m \times 1$ matrix (m rows and one column) $\begin{pmatrix} a_{11} \\ a_{21} \\ \cdots \\ a_{m1} \end{pmatrix}$.

A matrix of size $n \times n$ is said ***square*** matrix ***of n-th order***. In a square matrix A, the elements $a_{ii}\ (i = 1, 2, \ldots, n)$ are named its ***principal diagonal elements*** and form a ***principal diagonal.*** The elements $a_{i,n-i+1}\ (i = 1, 2, \ldots, n)$ are named a ***secondary diagonal elements*** and form a ***secondary diagonal*** of matrix A.

A matrix, all elements of which are equal to zero, is called the ***zero matrix*** and is denoted by the symbol **0**.

A matrix

$$\begin{pmatrix} a_{11} & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & a_{22} & 0 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & a_{n-1\,n} & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & a_{nn} \end{pmatrix}, \qquad (1.2)$$

which has all of its elements zero except the diagonal ones, i.e. $a_{ij} = 0$ for all $i \neq j$, is called the ***diagonal matrix***.

In a special case $a_{ii} = 1$ ($i = 1, 2, ..., n$) the diagonal matrix is called the ***identity matrix*** of order $n \times n$ and denoted by

$$E = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 1 \end{pmatrix}. \qquad \textbf{(1.3)}$$

A square matrix A whose elements satisfy $a_{ij} = 0$ for all $i > j$, is called ***an upper triangular matrix***, i.e.,

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a_{nn} \end{pmatrix}. \qquad \textbf{(1.5)}$$

A ***lower triangular matrix*** can be defined in a similar fashion, i.e. $a_{ij} = 0$ for all $i < j$. A diagonal matrix (1.2) is both an upper triangular matrix and a lower triangular matrix.

1.2. Matrix arithmetic

Equality of matrices. Two matrices A and B are said to be ***equal*** if they have the same size and corresponding elements are equal. That is, $A = (a_{ij})$ $(i = 1, 2, \dots, m; j = 1, 2, \dots, n)$ and $B = (b_{ij})$ $(i = 1, 2, \dots, m; j = 1, 2, \dots, n)$ and $a_{ij} = b_{ij}$ for $i = 1, 2, \dots, m$, $j = 1, 2, \dots, n$.

Addition of matrices. If $A=(a_{ij})$ and $B=(b_{ij})$ are two $m\times n$ matrices, their ***sum*** $A+B$ is an $m\times n$ matrix obtained by adding corresponding elements of A and B. Thus $A+B=(a_{ij})+(b_{ij})=(a_{ij}+b_{ij})$ for $i=1, 2, \dots, m,$ $j=1, 2, \dots, n.$

Subtraction of matrices. Matrix ***subtraction*** is defined for two matrices $A=(a_{ij})$ and $B=(b_{ij})$ of the same size, in the usual way. That is

$A-B=(a_{ij})-(b_{ij})=(a_{ij}-b_{ij})$ $(i=1, 2, \dots, m; j=1, 2, \dots, n)$.

Remark. Two matrices of the same order are said to be ***conformable*** for addition and subtraction. Addition and subtraction are not defined for matrices which are not conformable.

Scalar multiplication of a matrix. If $A=(a_{ij})$ is an $m\times n$ matrix and λ is a number (***scalar***), then λA is a matrix obtained by multiplying *all* elements of A by λ; that is $\lambda A=\lambda(a_{ij})=(\lambda a_{ij})$ $(i=1, 2, \dots, m; j=1, 2, \dots, n)$.

Therefore, $-A=(-1)A=(-a_{ij})$.

Example 1.1. $A=\begin{pmatrix}-3 & 4\\ -1 & 2\end{pmatrix}$, $B=\begin{pmatrix}2 & 5\\ 1 & -2\end{pmatrix}$.

$$3A+2B=\begin{pmatrix}-9 & 12\\ -3 & 6\end{pmatrix}+\begin{pmatrix}4 & 10\\ 2 & -4\end{pmatrix}=\begin{pmatrix}-9+4 & 12+10\\ -3+2 & 6-4\end{pmatrix}=\begin{pmatrix}-5 & 22\\ -1 & 2\end{pmatrix}.$$

The matrix operations of addition, subtraction and scalar multiplication satisfy the usual laws of arithmetic (in what follows, λ and μ will be arbitrary scalars and A, B, C are matrices as assumed to be conformable).

1. $A+B=B+A.$
2. $(A+B)+C=A+(B+C).$
3. $0+A=A.$ **(1.6)**
4. $A+(-A)=0.$
5. $\lambda(A\pm B)=\lambda A\pm\lambda B.$

6. $(\lambda \pm \mu) A = \lambda A \pm \mu A$.

7. $\lambda(\mu A) = (\lambda\mu) A$.

8. $1A = A,\ 0A = 0$.

9. $\lambda A = 0 \ \Rightarrow\ \lambda = 0$ or $A = 0$.

Matrix product. Let $A = (a_{ij})$ be a matrix of size $m \times n$ and $B = (b_{jk})$ be a matrix of size $n \times p$. Then product AB of matrices A and B is the $m \times p$ matrix $C = (c_{ik})$ whose (i, k)-th element is defined by the formula

$$c_{ik} = a_{i1}b_{1k} + a_{i2}b_{2k} + \dots + a_{in}b_{nk} = \sum_{j=1}^{n} a_{ij}b_{jk}, \tag{1.7}$$

where $i = 1, 2, \dots, m,\ k = 1, 2, \dots, p$.

Remark. The product AB is defined only when the number of columns of A is equal to the number of rows of B. If this is the case, A is said to be ***conformable*** to B for multiplication. If A is conformable to B, then B is not necessarily conformable to A.

Example 1.2. $A = \begin{pmatrix} 5 & -1 & 3 \\ 2 & -2 & 1 \\ 1 & 3 & -3 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 4 \\ -2 & -4 \\ 3 & -1 \end{pmatrix},$

$$AB = \begin{pmatrix} 5\times 1 + (-1)\times(-2) + 3\times 3 & 5\times 4 + (-1)\times(-4) + 3\times(-1) \\ 2\times 1 + (-2)\times(-2) + 1\times 3 & 2\times 4 + (-2)\times(-4) + 1\times(-1) \\ 1\times 1 + 3\times(-2) + (-3)\times 3 & 1\times 4 + 3\times(-4) + (-3)\times(-1) \end{pmatrix} = \begin{pmatrix} 16 & 21 \\ 9 & 15 \\ -14 & -5 \end{pmatrix}.$$

Matrix multiplication obeys associative and distributive laws:

1. $(AB)C = A(BC)$ if A, B, C are $m \times n$, $n \times p$, $p \times q$ matrices respectively.

2. $\lambda(AB) = (\lambda A)B = A(\lambda B)$.

3. $(A + B)C = AC + BC$ if A and B are matrices of size $m \times n$ and C is $n \times p$ matrix ("right distributivity").

4. $A\ (B + C) = AB + AC$ if B and C are matrices of size $m \times n$ and A is $p \times m$ matrix ("left distributivity").

5. $A\,E = E\,A = A$ if A is any $n \times n$ matrix, that is, E is the ***multiplicative identity*** for the set of $n \times n$ matrices.

6. In general, $AB \neq BA$, i.e. even if BA is defined, it is not necessarily equal to AB. Therefore in general, $AB = \mathbf{0}$ does not imply $A = \mathbf{0}$ or $B = \mathbf{0}$, and $AB = AC$ does not necessarily imply $B = C$.

1.3. Transpose matrix

Let A is the matrix (1.1) of size $m \times n$. Then the matrix of size $n \times m$ obtained by interchanging the rows and columns matrix A is called the ***transpose*** of A and is denoted by A^T.

That is $a_{ij}^T = a_{ji}$ or

$$A^T = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}^T = \begin{pmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{m2} \\ \dots & \dots & \dots & \dots \\ a_{1n} & a_{2n} & \dots & a_{mn} \end{pmatrix}. \qquad \textbf{(1.8)}$$

It is easily shown that

$$\left(A^T\right)^T = A,$$

$$(\lambda A)^T = \lambda A^T,$$

$$(A + B)^T = A^T + B^T, \qquad \textbf{(1.9)}$$

$$(AB)^T = B^T A^T.$$

2. Determinants

Historical reference. Historically, determinants were considered before matrices. In the seventh chapter of above-mentioned Chinese text (see hist. ref. to 1), “Too much and not enough”, the concept of a determinant first appears, almost 2000 years before its invention by the Japanese mathematician Seki Kowa in 1683.

Originally, a determinant was defined as a property of a system of linear equations. The determinant “determines” whether the system has a unique solution (which occurs precisely if the determinant is non-zero). In this sense, two-by-two determinants were considered by Cardano at the end of 16^{th} century. German Gottfried Leibniz (who is also credited with the invention of differential calculus, separately from but simultaneously with Isaac Newton) developed the theory of determinants in 1693. Following him Cramer developed the theory further, treating the subject in relation to sets of equations, and presented Cramer’s rule in 1750. The recurrent law was first announced by Bezout in 1764.

It was Vandermonde (1771) who first recognized determinants as independent functions. Laplace (1772) gave the general method of expanding a determinant in terms of its complementary minors: Vandermonde had already given a special case. Immediately following, Lagrange (1773) treated determinants of the second and third order. Lagrange was the first to apply determinants to questions outside elimination theory; he proved many special cases of general identities.

Carl Friedrich Gauss and Wilhelm Jordan developed Gauss-Jordan elimination in the 1800s. Gauss (1801) made the next advance. Like Lagrange, he made much use of determinants in the theory of numbers. He introduced the word ***determinants*** (Laplace had used ***resultant***), though not in the present signification, but rather as applied to the discriminant of a quantic. Gauss also arrived at the notion of reciprocal (inverse) determinants, and came very near the multiplication theorem.

The next contributor of importance is Binet (1811, 1812), who formally stated the theorem relating to the product of two matrices of m columns and n rows, which for the special case of $m = n$ reduces to the multiplication theorem. On the same day (Nov. 30, 1812) that Binet presented his paper to the Academy, Cauchy also presented one on the subject. In this he used the word ***determinant*** in its present sense, summarized and simplified what was then known on the subject, improved the notation, and gave the multiplication theorem with a proof more satisfactory than Binet’s. With him begins the theory in its generality.

1.4. Basic definitions

Every square matrix A of size $n \times n$ can be associated with a unique function depending on elements of A and possessing some specific properties. This function is called the ***determinant of a matrix A***.

The common definition of determinant (the so-called ***Leibniz formula***) is sufficiently complicated and here is not given. In addition in overwhelming majority of cases this definition is useless for practical computations. Therefore we shall be restricted to definitions of a determinant for partial cases of matrixes $\mathbf{2 \times 2}$ and $\mathbf{3 \times 3}$, and as definition of a determinant in a common case of a matrix $n \times n$ we shall conditionally accept one of ways of its computation (so-called ***Laplace expansion***).

The determinant of matrix A denoted by $\det A$ or $|A|$. Second notation is also used to denote the absolute value. However, the absolute value of a matrix is, in general, not defined. Thus, the notation for determinant by vertical bars on both sides of the matrix is frequently used.

Determinant of the second order. If A is a $\mathbf{2 \times 2}$ matrix, that is expression $a_{11}a_{22} - a_{12}a_{21}$ is called the ***determinant of*** $\mathbf{2 \times 2}$ ***matrix*** or ***determinant of the second order***, so

$$\det A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}. \qquad \mathbf{(1.10)}$$

<u>**Example 1.3.**</u> Find a determinant $\begin{vmatrix} 3 & -2 \\ 4 & -5 \end{vmatrix} = 3 \cdot (-5) - 4 \cdot (-2) = -7.$

Determinant of the third order. If A is a $\mathbf{3 \times 3}$ matrix, that is expression $a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} -$
$- a_{13}a_{22}a_{31}$ is called the ***determinant of*** $\mathbf{3 \times 3}$ ***matrix*** or ***determinant of the third order***, so

$$\det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} -$$

$$- a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}. \tag{1.11}$$

Minor. Let $A = (a_{ij})$ be a $n \times n$ matrix. The ***minor*** M_{ij} ***corresponding to the element*** a_{ij} (or simply ***minor of*** a_{ij}) is the determinant of the $(n-1)\times(n-1)$ submatrix of A formed by deleting the i-th row and j-th column of A containing the element a_{ij}.

Cofactor. The ***cofactor corresponding to the element*** a_{ij} (or simply ***cofactor of*** a_{ij}) is

$$A_{ij} = (-1)^{i+j} M_{ij}. \tag{1.12}$$

<u>Remark.</u> The expression $(-1)^{i+j}$ obeys the chess-board pattern of signs:

$$\begin{pmatrix} + & - & + & \dots \\ - & + & - & \dots \\ + & - & + & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}.$$

Determinant of n-th order. Let $A = (a_{ij})$ be a matrix of size $n \times n$ ($n > 2$). The determinant of A (or ***determinant of n-th order***) is the sum of the entries in any row or any column multiplied by their respective cofactors.

Applying this definition to find a determinant is called ***expanding by cofactors***.

Expanding by the i-th row called ***i-th row Laplace expansion***

$$\det A = a_{i1}A_{i1} + a_{i2}A_{i2} + \dots + a_{in}A_{in} = \sum_{j=1}^{n} a_{ij}A_{ij} \tag{1.13}$$

and expanding by the j-th column called ***j-th column Laplace expansion***

$$\det\ A = a_{1j}A_{1j} + a_{2j}A_{2j} + \ldots + a_{nj}A_{nj} = \sum_{i=1}^{n} a_{ij}A_{ij}. \qquad \textbf{(1.14)}$$

Remark. From last definitions follows, as it is easy to see, that the finding of a determinant of n-th order is reduced to a finding of n determinants of (n-1)-th order. Therefore Laplace expansion is efficient for computation of determinants of relatively small matrices. In general, determinants can be computed using Gaussian elimination. Laplace expansion is of theoretical interest as one of several ways to view the determinant, as well as of practical use in determinant computation.

If A is a 3×3 matrix, then, using the first- row Laplace expansion, we receive the definition of ***determinant of a third order*** in the form of

$$\det\ A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}A_{11} - a_{12}A_{12} + a_{13}A_{13} =$$

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} =$$

$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} -$$

$$- a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}. \qquad \textbf{(1.15)}$$

Note. A convenient methods for calculating determinants of the third order are ***mnemonic Sarrus rules***. First rule is known as ***a triangles rule***. Schemes of these rules are illustrated by Fig. 1.1 and Fig. 1.2. The conditional lines connecting elements of determinants designate corresponding products. The products of the elements connecting by dotted lines on Fig. 1.2 we take with the sign " – ".

Example 1.4. Find a determinant with use of a triangles rule

$$\begin{vmatrix} 2 & 4 & -1 \\ -3 & 5 & 4 \\ 1 & 0 & -3 \end{vmatrix} = 2\cdot5\cdot(-3)+4\cdot4\cdot1+(-1)\cdot(-3)\cdot0-(-1)\cdot5\cdot1-$$

$$-4\cdot(-3)\cdot(-3)-2\cdot4\cdot0 = -45.$$

Remark. For reduction of calculations it is best to choose a row or column with the greatest quantity of zeros, so we use second-column Laplace expansion:

$$\begin{vmatrix} 2 & 4 & -1 \\ -3 & 5 & 4 \\ 1 & 0 & -3 \end{vmatrix} = -4\cdot\begin{vmatrix} -3 & 4 \\ 1 & -3 \end{vmatrix} + 5\cdot\begin{vmatrix} 2 & -1 \\ 1 & -3 \end{vmatrix} =$$

$$= -4\cdot(9-4) + 5\cdot(-6+1) = -20 - 25 = -45.$$

As we see, the received result has coincided with previous.

The first Sarrus rule (triangles rule)

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} =$$

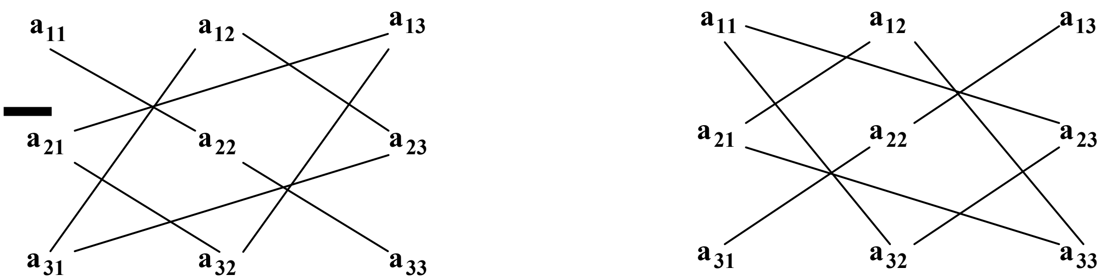


Fig. 1.1

The second Sarrus rule

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} =$$

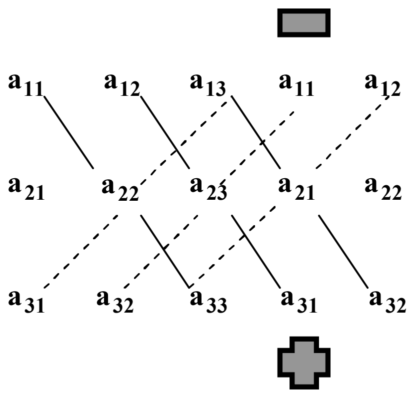


Fig. 1.2

1.5. Properties of determinants

Property 1. A matrix and its transpose have equal determinants; that is

$$\det A = \det(A^T). \qquad \textbf{(1.16)}$$

Remark. As it follows from this property the rows and the columns of a determinant are equal in rights. Therefore all properties which further we shall formulate with respect to rows, are correct and for columns.

Property 2. If any two rows of the determinant are interchanged, the determinant changes the sign.

Property 3. Let B is the matrix received from the matrix A by multiplying *one* row with the number λ, then

$$\det B = \lambda \det A. \qquad \textbf{(1.17)}$$

For example

$$\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2j} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \lambda a_{i1} & \lambda a_{i2} & \dots & \lambda a_{ij} & \dots & \lambda a_{in} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nj} & \dots & a_{nn} \end{vmatrix} = \lambda \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2j} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{i1} & a_{i2} & \dots & a_{ij} & \dots & a_{in} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nj} & \dots & a_{nn} \end{vmatrix}.$$

Property 4. The determinant is a linear function of each row.

For example

$$\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2j} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ b_{i1}+c_{i1} & b_{i2}+c_{i2} & \dots & b_{ij}+c_{ij} & \dots & b_{in}+c_{in} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nj} & \dots & a_{nn} \end{vmatrix} =$$

$$= \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2j} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ b_{i1} & b_{i2} & \dots & b_{ij} & \dots & b_{in} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nj} & \dots & a_{nn} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2j} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ c_{i1} & c_{i2} & \dots & c_{ij} & \dots & c_{in} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nj} & \dots & a_{nn} \end{vmatrix}. \quad (1.18)$$

Note. This property remains correct for any number of summands.

Property 5. If even one row is a linear combination of several another rows, the determinant is equal to zero.

In particular, determinant *is equal to zero* if

- even one row of a determinant is zero;
- even two rows of a determinant are equal;
- even two any rows are proportional.

Property 6. If any row of a determinant is added to any linear combination of several *another* rows, that the value of a determinant will not change.

In particular, determinant *will not change* if

- one any row is added to another row;
- if a multiple of any row is added to another row.

For example, if $i \neq k$, then

$$\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2j} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{i1} + \lambda a_{k1} & a_{i2} + \lambda a_{k2} & \dots & a_{ij} + \lambda a_{kj} & \dots & a_{in} + \lambda a_{kn} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nj} & \dots & a_{nn} \end{vmatrix} =$$

$$= \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2j} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{i1} & a_{i2} & \dots & a_{ij} & \dots & a_{in} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nj} & \dots & a_{nn} \end{vmatrix}. \qquad \textbf{(1.19)}$$

Properties of determinants are useful for their simplifying and numerically evaluating.

One of the simplest determinants to evaluate is that of an upper triangular matrix (1.4), i.e. if $A = (a_{ij})$, where $a_{ij} = 0$ if $i > j$, then

$$\det A = a_{11} \cdot a_{22} \cdot \ldots \cdot a_{nn}. \qquad \textbf{(1.20)}$$

Note. If A is a lower triangular matrix or in important special case when A is a diagonal matrix (1.2), equation (1.20) remains true.

In particular, as it easy to see,

$$\det E = 1 \cdot 1 \cdot \ldots \cdot 1 = 1. \qquad \textbf{(1.21)}$$

To evaluate a determinant numerically, it is advisable to reduce the matrix to *row-echelon form*, recording any sign changes caused by row interchanges, together with any factors taken out of a row, as in the following example.

Example 1.5. Find the determinant

$$\begin{vmatrix} 1 & 1 & 2 & 1 \\ 3 & 1 & 4 & 5 \\ 7 & 6 & 1 & 2 \\ 1 & 1 & 3 & 4 \end{vmatrix}.$$

Solution. Using notation of row operations (where R_i denote i-th row, $\to$ denotes the change of a row, $\leftrightarrow$ denotes the interchange of rows), we obtain (corresponding operations are explained in the braces)

$$\begin{vmatrix} 1 & 1 & 2 & 1 \\ 3 & 1 & 4 & 5 \\ 7 & 6 & 1 & 2 \\ 1 & 1 & 3 & 4 \end{vmatrix} = \{R_2 \to -3R_1 + R_2, R_3 \to -7R_1 + R_3, R_4 \to -R_1 + R_4\} =$$

$$= \begin{vmatrix} 1 & 1 & 2 & 1 \\ 0 & -2 & -2 & 2 \\ 0 & -1 & -13 & -5 \\ 0 & 0 & 1 & 3 \end{vmatrix} = \{R_3 \leftrightarrow R_2\} = -\begin{vmatrix} 1 & 1 & 2 & 1 \\ 0 & -1 & -13 & -5 \\ 0 & -2 & -2 & 2 \\ 0 & 0 & 1 & 3 \end{vmatrix} =$$

$$= \{R_3 \to -2R_2 + R_3\} = -\begin{vmatrix} 1 & 1 & 2 & 1 \\ 0 & -1 & -13 & -5 \\ 0 & 0 & 24 & 12 \\ 0 & 0 & 1 & 3 \end{vmatrix} = \{R_4 \to -\frac{1}{24}R_3 + R_4\} =$$

$$= -\begin{vmatrix} 1 & 1 & 2 & 1 \\ 0 & -1 & -13 & -5 \\ 0 & 0 & 24 & 12 \\ 0 & 0 & 0 & 5/2 \end{vmatrix} = -1 \cdot (-1) \cdot 24 \cdot \frac{5}{2} = 60.$$

In addition we'll consider theorems which will be useful further.

Theorem ***(determinants multiplication).*** Determinant of the product of several matrices of the same order is equal to product of determinants of these matrices.

In particular, for two matrices we have

$$\det(AB) = (\det A)(\det B). \qquad \textbf{(1.22)}$$

Corollary. It is easy to see that, as it follows from definition of a scalar multiplication of a matrix and from (1.22), $\det(\lambda E) = \lambda^n$ and thus

$$\det(\lambda A) = \det(\lambda E \cdot A) = \lambda^n \det A.$$

Theorem. The sum of the entries in any row (column) of matrix A multiplied by cofactors of respective entries in *other* row (column) is equal to zero, namely

$$\sum_{j=1}^{n} a_{ij}A_{kj} = 0 \text{ if } i \neq k \quad \left(\sum_{i=1}^{n} a_{ij}A_{ik} = 0 \text{ if } j \neq k \right). \qquad \textbf{(1.23)}$$

3. Solving systems of linear algebraic equations

1.6. The inverse of a square matrix

A square matrix A of size $n\times n$ is called ***invertible*** or ***nonsingular*** if there exist a square matrix A^{-1} of the same order such that

$$AA^{-1} = E = A^{-1}A, \qquad \textbf{(1.24)}$$

where E denotes the $n\times n$ identity matrix and the multiplication used is ordinary matrix multiplication. If this is the case, then the matrix A^{-1} is uniquely determined by A and is called the ***inverse*** of A.

A square matrix that is not invertible is called ***singular*** or ***degenerate***. ***Matrix inversion*** is the process of finding the matrix A^{-1} that satisfies the prior equation for a given invertible matrix A.

If A is an $n\times n$ matrix, the ***adjugate*** or ***adjoint*** of A, denoted by A^* or adj A, is the transpose of the matrix of cofactors. Hence

$$A^* = \begin{pmatrix} A_{11} & A_{21} & \dots & A_{n1} \\ A_{12} & A_{22} & \dots & A_{n2} \\ \dots & \dots & \dots & \dots \\ A_{1n} & A_{2n} & \dots & A_{nn} \end{pmatrix}. \qquad \textbf{(1.25)}$$

Remark. The adjugate is a matrix which plays a role similar to the inverse of a matrix; it can however be defined for any square matrix without the need to perform any divisions.

As a consequence of Laplace's expansion (1.13) for the determinant of an $n \times n$ matrix A and theorem (1.23), we have

$$A A^* = A^* A = \begin{pmatrix} \det A & 0 & \dots & 0 \\ 0 & \det A & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \det A \end{pmatrix} = (\det A)\, E. \qquad \textbf{(1.26)}$$

From here and from (1.24) follows, that if $\det A \neq 0$, then the inverse A^{-1} exists and is given by ***formula for inverse***

$$A^{-1} = \frac{1}{\det A} A^*. \qquad \textbf{(1.27)}$$

Of course, if A^{-1} exists, then, as it follows from (1.22) and (1.26),

$\det(A A^{-1}) = (\det A)(\det A^{-1}) = \det E = 1 \neq 0$. Therefore

$$\det A^{-1} = \frac{1}{\det A}. \qquad \textbf{(1.28)}$$

Thus from all aforesaid follows one of the most important results in matrix algebra, namely: **a matrix A is invertible if and only if its determinant is nonzero.**

Addition. It is easy to prove that the inverse of an invertible matrix A is itself invertible and is equal to the original matrix, i.e.

$$\left(A^{-1}\right)^{-1} = A,$$

and the inverse of the transpose is the transpose of the inverse

$$\left(A^T\right)^{-1} = \left(A^{-1}\right)^T.$$

Also it is shown, that if A and B are square matrices of the same order with inverses A^{-1} and B^{-1} respectively, then

$$\left(AB\right)^{-1} = B^{-1} A^{-1}. \qquad \textbf{(1.29)}$$

Remark. In the definition of an invertible matrix $\boldsymbol{A}$, we used both $\boldsymbol{A}\boldsymbol{A}^{-1}$ and $\boldsymbol{A}^{-1}\boldsymbol{A}$ to be equal to the identity matrix. In fact, we need only one of the two. In other words, for a matrix $\boldsymbol{A}$, if there exists a matrix $\boldsymbol{A}^{-1}$ such that $\boldsymbol{A}\boldsymbol{A}^{-1}=\boldsymbol{E}$, then $\boldsymbol{A}$ is invertible and $\boldsymbol{A}^{-1}$ is inverse of $\boldsymbol{A}$.

Example 1.6. Find the inverse of $A=\begin{pmatrix} 3 & 2 & 1 \\ 2 & -1 & 1 \\ 1 & 5 & 0 \end{pmatrix}$.

Solution. Calculate $\det A$ and cofactors to elements of A: $\det A = -2$,

$A_{11} = -5,\quad A_{12} = 1,\quad A_{13} = 11,\quad A_{21} = 5,\quad A_{22} = -1,\quad A_{23} = -13,$

$A_{31} = 3,\quad A_{32} = -1,\quad A_{33} = -7.$

Then adjugate of A is $A^{*} = \begin{pmatrix} -5 & 5 & 3 \\ 1 & -1 & -1 \\ 11 & -13 & -7 \end{pmatrix}$ and we find the inverse of A:

$$A^{-1} = \frac{1}{\det A} A^{*} = -\frac{1}{2}\begin{pmatrix} -5 & 5 & 3 \\ 1 & -1 & -1 \\ 11 & -13 & -7 \end{pmatrix} = \begin{pmatrix} 5/2 & -5/2 & -3/2 \\ -1/2 & 1/2 & 1/2 \\ -11/2 & 13/2 & 7/2 \end{pmatrix}.$$

Note. This way is efficient to calculate the inverse of *small* matrices (since this method is essentially recursive, it becomes inefficient for large matrices).

1.7. The concept of system of linear algebraic equations

A system of $\boldsymbol{n}$ linear algebraic equations in $\boldsymbol{n}$ unknowns $x_1, x_2, \ldots, x_n$ is a family of respective equations and can be written in unfolded form as

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1, \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2, \\ &\ldots\ldots\ldots\ldots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n &= b_n, \end{aligned} \tag{1.30}$$

where a_{ij} $(i, j = 1, 2, \ldots, n)$ are called ***coefficients*** of a system, $b_1, b_2, \ldots, b_n$ are called ***free terms***. If at least one free term is nonzero then system is called ***nonhomogeneous***. Otherwise, i.e. if all free terms simultaneously are equal to zero, the system is called ***homogeneous***.

If exists a set of n values $x_1, x_2, \ldots, x_n$ which satisfy *each* of the equations simultaneously, i.e. turn *all* equations of system (1.30) into true identities, then it is called a ***solution*** of system. A system of equations is called ***compatible*** or ***consistent*** if it has at least one solution. A system that has no solution is called ***incompatible*** or ***inconsistent***.

Note that the above system can be written concisely as

$$\sum_{j=1}^{n} a_{ij} x_j = b_i \quad \text{for} \quad i = 1, 2, \ldots, n. \tag{1.31}$$

The square matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \tag{1.32}$$

is called the ***coefficient matrix*** of the system. The column vector

$X = \begin{pmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{pmatrix}$ is called the ***vector of unknowns*** and the column vector $B = \begin{pmatrix} b_1 \\ b_2 \\ \cdots \\ b_n \end{pmatrix}$

is called the ***vector of free terms***. Then the system (1.30) can be rewritten in compact matrix form

$$A\,X = B. \tag{1.33}$$

1.8. Solving a system of equations using an inverse

If coefficient matrix A is an invertible matrix, the system of linear algebraic equations represented by (1.33) has a unique solution given by

$$X = A^{-1}B. \tag{1.34}$$

Example 1.7. Use an inverse matrix to solve the system

$$\begin{aligned} 3x_1 + 2x_2 + x_3 &= 5, \\ 2x_1 - x_2 + x_3 &= 6, \\ x_1 + 5x_2 \qquad &= -3. \end{aligned}$$

Solution. A coefficient matrix of set system is considered earlier in an example 1.6. Since its determinant is not zero, then the solution of system exists and is unique, namely

$$X = A^{-1}B = -\frac{1}{2}\begin{pmatrix} -5 & 5 & 3 \\ 1 & -1 & -1 \\ 11 & -13 & -7 \end{pmatrix}\begin{pmatrix} 5 \\ 6 \\ -3 \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} -4 \\ 2 \\ -2 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}.$$

Thus, the solution is $x_1 = -2,\ x_2 = -1,\ x_3 = 1.$

Substitution of the found values of unknowns into equations shows, that the solution is found truly.

Remark. Check of *all* equations is obligatory!

Geometrically, solving a system of linear equations in two (or three) unknowns is equivalent to determining whether or not a family of lines (or planes) has a common point or intersection.

1.9. Cramer's rule

To finish this section, we present an old (1750) method of solving a system of n equations in n unknowns called *Cramer's rule.* This method is not used in practice. However it has a theoretical use as it reveals explicitly how the solution depends on the coefficient matrix.

Theorem ***(Cramer's rule).*** The linear system $AX = B$ has an unique solution if and only if matrix A is invertible. In this case, the solution is given by the so-called ***Cramer's formulas***:

$$x_i = \frac{\Delta_i}{\Delta} \text{ for } i = 1, 2, \ldots, n, \tag{1.35}$$

where $\Delta = \det A$ is called the ***principal determinant*** of a system, and determinants Δ_i are obtained from Δ by replacing the ***i***-th column by the column B of free terms and are called ***auxiliary determinants***:

$$\Delta_i = \begin{vmatrix} a_{11} & a_{12} & \ldots & a_{1i-1} & b_1 & a_{1i+1} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2i-1} & b_2 & a_{2i+1} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{ni-1} & b_n & a_{ni+1} & \ldots & a_{nn} \end{vmatrix}.$$

Proof. Suppose the principal determinant $\Delta \neq 0$. Then inverse A^{-1} exists and is given by (1.27) and the system has unique solution

$$\begin{pmatrix} x_1 \\ x_2 \\ \ldots \\ x_i \\ \ldots \\ x_n \end{pmatrix} = A^{-1} \begin{pmatrix} b_1 \\ b_2 \\ \ldots \\ b_i \\ \ldots \\ b_n \end{pmatrix} = \frac{1}{\Delta} \begin{pmatrix} A_{11} & A_{21} & \ldots & A_{n1} \\ A_{12} & A_{22} & \ldots & A_{n2} \\ \ldots & \ldots & \ldots & \ldots \\ A_{1i} & A_{2i} & \ldots & A_{ni} \\ \ldots & \ldots & \ldots & \ldots \\ A_{1n} & A_{2n} & \ldots & A_{nn} \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ \ldots \\ b_i \\ \ldots \\ b_n \end{pmatrix} =$$

$$= \frac{1}{\Delta} \begin{pmatrix} b_1 A_{11} + b_2 A_{21} + \ldots + b_n A_{n1} \\ b_1 A_{12} + b_2 A_{22} + \ldots + b_n A_{n2} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ b_1 A_{1i} + b_2 A_{2i} + \ldots + b_n A_{ni} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ b_1 A_{1n} + b_2 A_{2n} + \ldots + b_n A_{nn} \end{pmatrix}.$$

However the ***i***-th component of the last vector is ***i***-column expansion of Δ_i. Hence

$$\begin{pmatrix} x_1 \\ x_2 \\ \cdots \\ x_i \\ \cdots \\ x_n \end{pmatrix} = \frac{1}{\Delta}\begin{pmatrix} \Delta_1 \\ \Delta_2 \\ \cdots \\ \Delta_i \\ \cdots \\ \Delta_n \end{pmatrix} = \begin{pmatrix} \Delta_1/\Delta \\ \Delta_2/\Delta \\ \cdots\cdots \\ \Delta_i/\Delta \\ \cdots\cdots \\ \Delta_n/\Delta \end{pmatrix}.$$

Example 1.8. Use Cramer's rule to solve the system given in example 1.7.

Solution. Since the principal determinant of a system $\Delta = -2$ and is not zero, then the solution exists and is unique. Therefore we can apply Cramer's rule.

Find auxiliary determinants:

$$\Delta_1 = \begin{vmatrix} 5 & 2 & 1 \\ 6 & -1 & 1 \\ -3 & 5 & 0 \end{vmatrix} = -4, \ \Delta_2 = \begin{vmatrix} 3 & 5 & 1 \\ 2 & 6 & 1 \\ 1 & -3 & 0 \end{vmatrix} = 2, \ \Delta_3 = \begin{vmatrix} 3 & 2 & 5 \\ 2 & -1 & 6 \\ 1 & 5 & -3 \end{vmatrix} =$$

$$= -2.$$

Then solution is $x_1 = \frac{\Delta_1}{\Delta} = 2$, $x_2 = \frac{\Delta_2}{\Delta} = -1$, $x_3 = \frac{\Delta_3}{\Delta} = 1$ and, as we see, has coincided with earlier found.

Addition. One important result is obtained in particular case, when linear system $AX = B$ is homogeneous, i.e. $B = 0$. Then if A is invertible, the system has only ***trivial solution*** $X = 0$. However if matrix A is noninvertible, then (in addition to trivial solution) homogeneous system will has also nonzero solution. From previous follows, that this will happen if and only if $\det A = 0$.

S e c t i o n 2

VECTORS

1. Introduction to vector algebra

2.1. Fundamental concepts and notations

There are quantities in physics characterized by magnitude only, such as time, temperature, mass and length. Such quantities are called ***scalars***. They are nothing more than real numbers and will be denoted as usual by letters *a, b, c,…* in ordinary type.

Other quantities characterized by both magnitude and direction, such as force, displacement, velocity and acceleration. To describe such quantities, we introduce the concept of a ***vector*** as a directed line segment (arrows) $\overrightarrow{\mathbf{MN}}$ from one point *M* called the ***initial*** or ***beginning point*** to another point *N* called the ***terminal*** or ***end point***. The direction of the arrow (the angle that it makes with some fixed directed line of the plane or space) is the direction of the vector, and the length of the arrow represents the ***magnitude*** or ***length*** of the vector.

We'll denote vectors as usually by letters with an arrow over them, i.e. $\vec{\mathbf{a}}, \vec{\mathbf{b}}, \vec{\mathbf{c}},\ldots$. Vector $\overrightarrow{\mathbf{MN}}$ (where it is assumed that the vector goes from *M* to *N*) also can be denoted by $\vec{\mathbf{a}}$ as in Fig. 2.1.

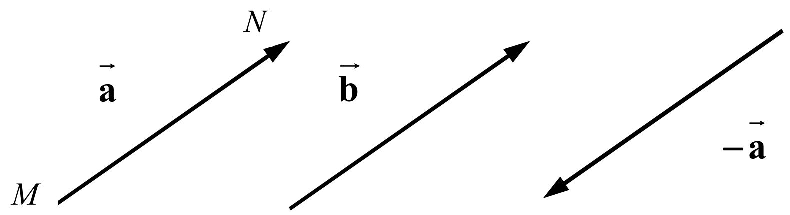


Fig. 2.1

According to this the magnitude (length) of a vector $\vec{\mathbf{a}}$ or $\overrightarrow{\mathbf{MN}}$ will denoted respectively by $\left|\vec{\mathbf{a}}\right|$ or $\left|\overrightarrow{\mathbf{MN}}\right|$.

Two vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ are said to be ***equal*** if they have the same magnitude and direction regardless of their initial points. Thus we write $\vec{\mathbf{a}} = \vec{\mathbf{b}}$.

A vector whose magnitude is that of $\vec{\mathbf{a}}$, but whose direction is opposite that of $\vec{\mathbf{a}}$, is called the ***negative*** of $\vec{\mathbf{a}}$ and is denoted $-\vec{\mathbf{a}}$ [see Fig. 2.1].

A vector, which has a magnitude of zero but its direction is not defined, is called the ***null*** or ***zero vector*** and is denoted by symbol $\vec{\mathbf{0}}$.

Unless indicated otherwise, a given vector has no fixed position in the plane or in the space and so may be moved parallel displacement at will. In particular, if $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ are two vectors [Fig. 2.2], they may be placed so as to have a common initial point M [Fig. 2.3] or so that the initial point of $\vec{\mathbf{b}}$ coincides with the terminal point of $\vec{\mathbf{a}}$ [Fig. 2.4].

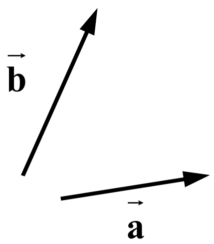


Fig. 2.2

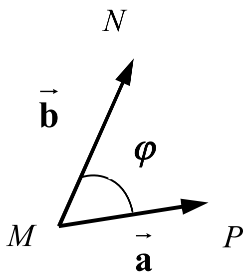


Fig. 2.3

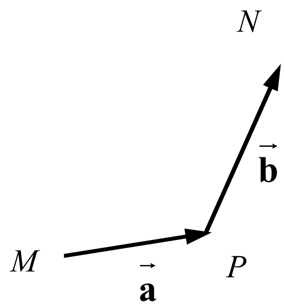


Fig. 2.4

The ***angle*** between two nonzero vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ is simply the angle between the directions of these vectors. If the vectors have a common initial point (so-called *standard* position), then the angle φ between them [Fig. 2.3] is the corresponding angle $0 \le \varphi \le 180^\circ$ (or $0 \le \varphi \le \pi$) between their respective standard position representatives. Further we'll also denote this angle by $(\vec{\mathbf{a}} \,\hat{,}\, \vec{\mathbf{b}})$.

2.2. Linear vector operations and their properties

Two basic linear vector operations are *scalar multiplication* (multiplying a vector by a number, i.e. scalar) and *vector addition (subtraction)*. These operations familiar in the algebra of numbers are, with suitable definition, capable of extension to an algebra of vectors. Also, as we'll see, these operations satisfy many properties similar to those for numbers.

Multiplication of a vector $\vec{\mathbf{a}}$ by a scalar λ produces a vector $\lambda\vec{\mathbf{a}}$ with magnitude $|\lambda|$ times the magnitude of $\vec{\mathbf{a}}$ and direction the same as or opposite to that of $\vec{\mathbf{a}}$ according as λ is positive or negative [Fig. 2.5]. A vector having unit magnitude is called ***unit*** vector. Therefore if $\vec{\mathbf{a}}$ is any nonzero vector, then the vector $\overrightarrow{\mathbf{a}^\circ} = \dfrac{\vec{\mathbf{a}}}{|\vec{\mathbf{a}}|}$ such that $\left|\overrightarrow{\mathbf{a}^\circ}\right| = 1$, is called a ***unit vector in the direction of*** $\vec{\mathbf{a}}$. Then $\vec{\mathbf{a}} = |\vec{\mathbf{a}}|\overrightarrow{\mathbf{a}^\circ}$**.** Unit vectors provide a way to represent the direction of any nonzero vector. Any vector in the direction of $\vec{\mathbf{a}}$, or the opposite direction, is a scalar multiple of this unit vector $\overrightarrow{\mathbf{a}^\circ}$.

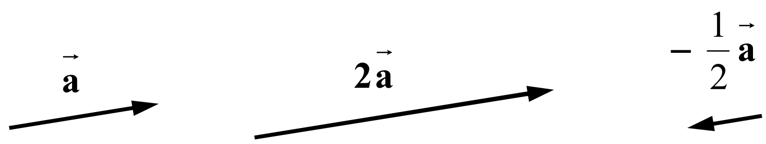


Fig. 2.5

The ***sum*** or ***resultant*** of the vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ is the vector $\vec{\mathbf{a}} + \vec{\mathbf{b}}$ which can be found in either of two equivalent ways:

1. By placing the initial point of $\vec{\mathbf{b}}$ on the terminal point of $\vec{\mathbf{a}}$ as in Fig. 2.6. Then the required sum is the vector $\overrightarrow{\mathbf{MN}}$ joining an initial point of $\vec{\mathbf{a}}$ to the terminal point of $\vec{\mathbf{b}}$. This procedure is called the ***triangle law*** for vector addition.

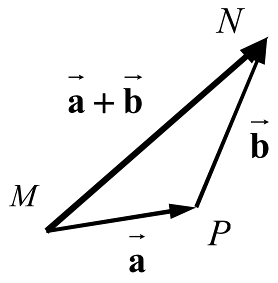


Fig. 2.6

2. By placing the initial point of $\vec{\mathbf{b}}$ on the initial point of $\vec{\mathbf{a}}$ and completing the parallelogram *MPQN* as in Fig. 2.7. The required sum is a diagonal *MQ* of the parallelogram. This way of vector addition has received the name of the ***parallelogram law***.

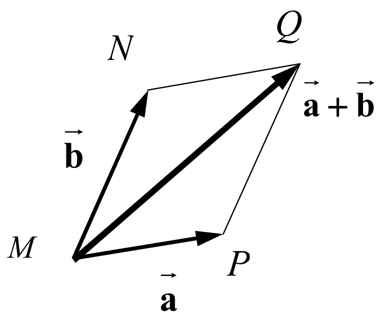
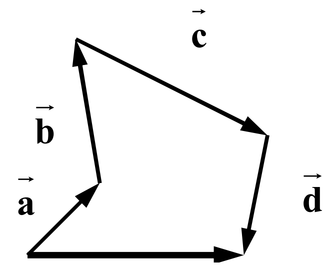


Fig. 2.7

Extension to sum of more than two vectors are immediate. Thus it is necessary to note, that the terminal point of a previous vector must coincides with an initial point of a following vector. Then the sum vector is the vector joining the initial point of the first vector to the terminal point of the last vector. For example, in Fig. 2.8 is shown how to obtain the sum of the vectors $\vec{\mathbf{a}}, \vec{\mathbf{b}}, \vec{\mathbf{c}}$ and $\vec{\mathbf{d}}$.

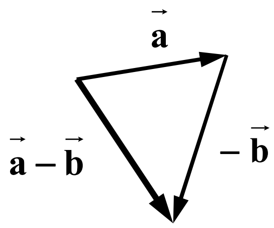


$\vec{a}+\vec{b}+\vec{c}+\vec{d}$

Fig. 2.8

The ***difference*** of the vectors $\vec{a}$ and $\vec{b}$ is the vector $\vec{a}-\vec{b}$ which can be defined in either of two equivalent ways:

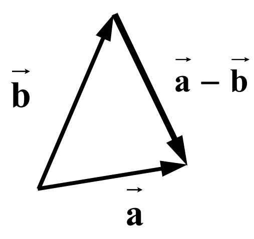
1. From the relation $\vec{a}-\vec{b}=\vec{a}+(-\vec{b})$, where addition is realized according to triangle law as in Fig. 2.9.



$\vec{a}-\vec{b}=\vec{a}+(-\vec{b})$

Fig. 2.9

2. As that vector which added to $\vec{b}$ (according to triangle law) gives $\vec{a}$, i.e. $\vec{b}+(\vec{a}-\vec{b})=\vec{a}$. This way is shown in Fig. 2.10. In other words, subtraction is defined as the inverse operation of addition.



$\vec{b}+(\vec{a}-\vec{b})=\vec{a}$

Fig. 2.10

As it easy to see the vectors $\vec{\mathbf{a}}+\vec{\mathbf{b}}$ and $\vec{\mathbf{a}}-\vec{\mathbf{b}}$ coincide with the diagonals of the parallelogram formed by vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ [Fig. 2.11].

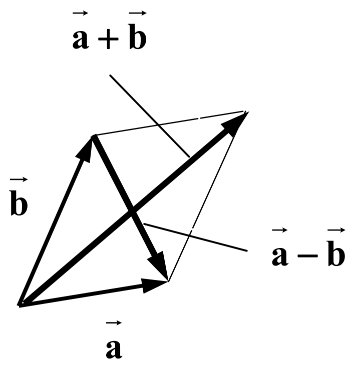


Fig. 2.11

If $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$ and $\vec{\mathbf{c}}$ are vectors, and λ and μ are scalars, then the following properties of linear vector operations are valid.

1. $\vec{\mathbf{a}}+\vec{\mathbf{b}}=\vec{\mathbf{b}}+\vec{\mathbf{a}}$ (*Commutative Law for Addition*).

2. $\vec{\mathbf{a}}+(\vec{\mathbf{b}}+\vec{\mathbf{c}})=(\vec{\mathbf{a}}+\vec{\mathbf{b}})+\vec{\mathbf{c}}$ (*Associative Law for Addition*).

3. $\lambda(\mu\,\vec{\mathbf{a}})=(\lambda\,\mu)\vec{\mathbf{a}}=\mu(\lambda\,\vec{\mathbf{a}})$ (*Associative Law for Multiplication*).

4. $(\lambda+\mu)\vec{\mathbf{a}}=\lambda\,\vec{\mathbf{a}}+\mu\,\vec{\mathbf{a}}$ (*Distributive Law*).

5. $\lambda(\vec{\mathbf{a}}+\vec{\mathbf{b}})=\lambda\,\vec{\mathbf{a}}+\lambda\,\vec{\mathbf{b}}$ (*Distributive Law*).

Note that in these laws only multiplication of a vector by one or more scalars is defined. The products of vectors will be defined later.

6. $\vec{\mathbf{a}}+\vec{\mathbf{0}}=\vec{\mathbf{a}}$.

7. $\vec{\mathbf{a}}+(-\vec{\mathbf{a}})=\vec{\mathbf{0}}$.

8. $\lambda\vec{\mathbf{0}}=\vec{\mathbf{0}}$, $-\vec{\mathbf{0}}=\vec{\mathbf{0}}$, $0\vec{\mathbf{a}}=\vec{\mathbf{0}}$.

<u>**Example 2.1.**</u> Find the lengths of diagonals of the parallelogram formed by vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ if $\left|\vec{\mathbf{a}}\right|=8$, $\left|\vec{\mathbf{b}}\right|=6$, $\varphi=(\vec{\mathbf{a}}\,\widehat{,}\,\vec{\mathbf{b}})=60^\circ$.

Solution. Taking into account that the diagonals of the parallelogram coincide with the vectors $\vec{\mathbf{a}}+\vec{\mathbf{b}}$ and $\vec{\mathbf{a}}-\vec{\mathbf{b}}$ [see Fig. 2.11], we'll find the lengths of diagonals as $\left|\vec{\mathbf{a}}-\vec{\mathbf{b}}\right|$ and $\left|\vec{\mathbf{a}}+\vec{\mathbf{b}}\right|$. Using the law of cosines, we receive

$\left|\vec{\mathbf{a}}-\vec{\mathbf{b}}\right|^2=\left|\vec{\mathbf{a}}\right|^2+\left|\vec{\mathbf{b}}\right|^2-2\left|\vec{\mathbf{a}}\right|\left|\vec{\mathbf{b}}\right|\mathbf{cos}\,\varphi=8^2+6^2-2\cdot 6\cdot 8\cdot\frac{1}{2}=52$, hence $\left|\vec{\mathbf{a}}-\vec{\mathbf{b}}\right|=2\sqrt{13}\approx 7{,}21$. Also we know that $\left|\vec{\mathbf{a}}-\vec{\mathbf{b}}\right|^2+\left|\vec{\mathbf{a}}+\vec{\mathbf{b}}\right|^2=2\left(\left|\vec{\mathbf{a}}\right|^2+\left|\vec{\mathbf{b}}\right|^2\right)$.

Then $\left|\vec{\mathbf{a}}+\vec{\mathbf{b}}\right|^2=2\cdot(8^2+6^2)-52=148$, hence $\left|\vec{\mathbf{a}}+\vec{\mathbf{b}}\right|=2\sqrt{37}\approx 12{,}17$.

2.3. Vectors in rectangular coordinate system

Three vectors $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$ and $\vec{\mathbf{c}}$ not in the same plane (i.e. ***not coplanar***) and no two parallel, issuing from a common point are said to form a ***right-handed*** (or ***dextral***) ***system*** or ***triad*** if $\vec{\mathbf{c}}$ has the direction in which the right-threaded screw would move when rotated through the smaller angle (less than 180°) in the direction from $\vec{\mathbf{a}}$ to $\vec{\mathbf{b}}$, as in Fig. 2.12. Note that, as seen from a terminal point of $\vec{\mathbf{c}}$, the rotation through the smaller angle from $\vec{\mathbf{a}}$ to $\vec{\mathbf{b}}$ is counterclockwise.

If $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$ and $\vec{\mathbf{c}}$ are also unit mutually orthogonal vectors them usually designate by $\vec{\mathbf{i}},\vec{\mathbf{j}}$ and $\vec{\mathbf{k}}$ and say these vectors form ***right-handed orthonormalized basis***. Thus $\left|\vec{\mathbf{i}}\right|=\left|\vec{\mathbf{j}}\right|=\left|\vec{\mathbf{k}}\right|=1$ and $\vec{\mathbf{i}}\perp\vec{\mathbf{j}}\perp\vec{\mathbf{k}}$.

Let's choose a rectangular coordinate system *Oxyz* having equal units of measure on all axes. Let also the positive *Ox*, *Oy* and *Oz* axes having the

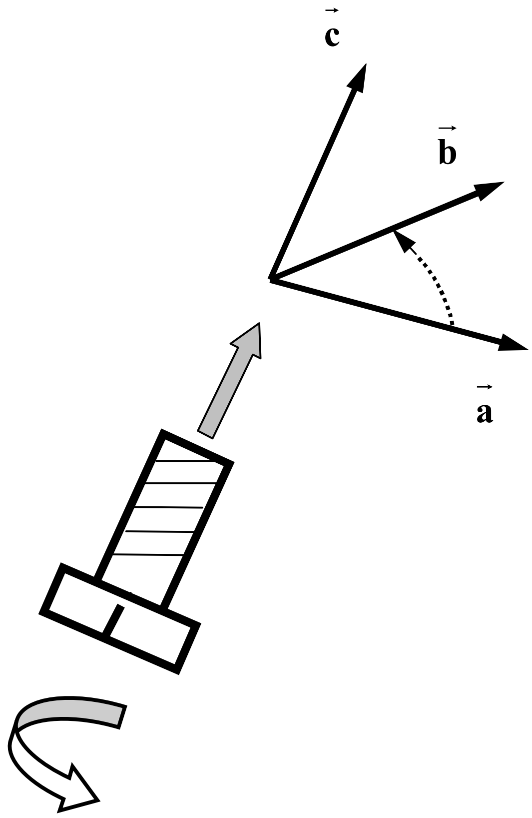


Fig. 2.12

direction of the vectors $\vec{\mathbf{i}}$, $\vec{\mathbf{j}}$, $\vec{\mathbf{k}}$ respectively as in Fig. 2.13. This system is called ***right-handed rectangular coordinate system*** and special vectors $\vec{\mathbf{i}}$, $\vec{\mathbf{j}}$, $\vec{\mathbf{k}}$ are often called ***coordinate*** vectors.

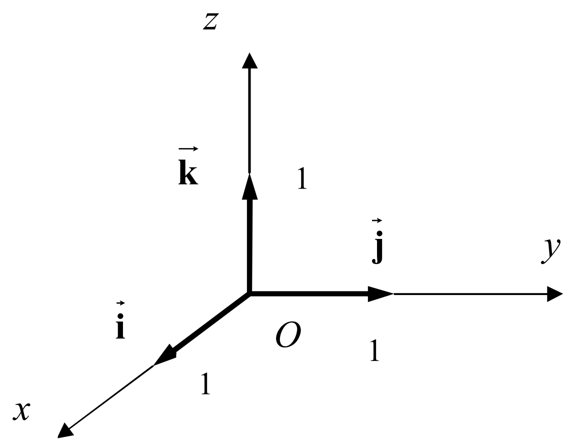


Fig. 2.13

Suppose we have an ordered triple (x, y, z) of real numbers [see Fig. 2.13]. The point in the space associated with this ordered triple is found as intersection point of three planes constructing perpendicular to the axes Ox, Oy and Oz through the points x, y, z respectively. There is exactly one point in the space thus associated with an ordered triple (x, y, z), and, conversely, each point in space determines by exactly one ordered triple (x, y, z) of real numbers. This procedure establishes a so-called ***one-to-one correspondence*** between ordered triples of real numbers and points in the space. Three numbers x, y and z, where (x, y, z) is the triple corresponding to the point M, are called, respectively, the ***x***, ***y***, ***z*** ***coordinates*** of M, and we write $M = (x, y, z)$ or $M(x, y, z)$. The only point $O(0, 0, 0)$ common to all three axes is called the ***origin***. The vector $\vec{\mathbf{r}} = \overrightarrow{\mathbf{OM}}$ joining the origin to point M is called the ***position vector*** or ***radius vector*** of M. We note $\vec{\mathbf{i}}$ is the position vector of the point $(1, 0, 0)$, $\vec{\mathbf{j}}$ is the position vector of $(0, 1, 0)$ and $\vec{\mathbf{k}}$ is the position vector of $(0, 0, 1)$.

Construct the parallelepiped as in Fig. 2.14. Using the linear vector operations and taking into account $\left|\vec{\mathbf{i}}\right| = \left|\vec{\mathbf{j}}\right| = \left|\vec{\mathbf{k}}\right| = 1$, we receive

$$\vec{\mathbf{r}} = x\vec{\mathbf{i}} + y\vec{\mathbf{j}} + z\vec{\mathbf{k}}. \qquad \textbf{(2.1)}$$

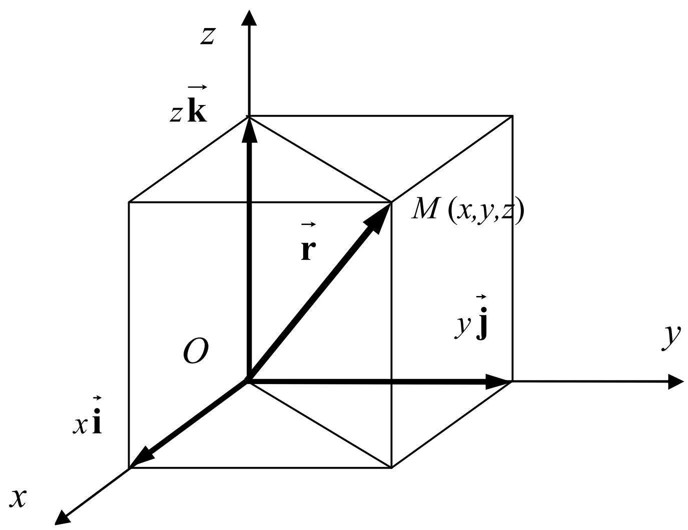


Fig. 2.14

Remark. The fact that the point with coordinates (x, y, z) is associated with the vector $\vec{\mathbf{r}}$ in this manner is shorthandedly indicated by writing $\vec{\mathbf{r}} = \{x, y, z\}$ (it is so-called ***component***

form of a vector). Strictly speaking this equation makes no sense; directed line segment cannot possible be a triple of real numbers, but this shorthand is usually clear and saves a lot of verbiage. Thus we frequently do not distinguish between points and their position vectors and say about one-to-one correspondence between them.

Let's call $x\vec{\mathbf{i}}$, $y\vec{\mathbf{j}}$ and $z\vec{\mathbf{k}}$ the ***vector components*** of $\vec{\mathbf{r}}$. As it follows from (2.1) the position vector of any point in space can be expressed as a linear combination of its vector components. The scalars *x*, *y* and *z* will be called the ***scalar components*** (or the ***x component***, ***y component*** and ***z component***, or simply the ***components*** [***coordinates***]) of $\vec{\mathbf{r}}$. Note that $\vec{\mathbf{0}} = 0\vec{\mathbf{i}} + 0\vec{\mathbf{j}} + 0\vec{\mathbf{k}}$.

If $M_1(x_1, y_1, z_1)$ and $M_2(x_2, y_2, z_2)$ [see Fig. 2.15], then the vector $\overrightarrow{\mathbf{M}_1\mathbf{M}_2}$, as it easy to see, is the difference of the position vectors $\overrightarrow{\mathbf{OM}_2}$ and $\overrightarrow{\mathbf{OM}_1}$, i.e. $\overrightarrow{\mathbf{M}_1\mathbf{M}_2} = \overrightarrow{\mathbf{OM}_2} - \overrightarrow{\mathbf{OM}_1} = x_2\vec{\mathbf{i}} + y_2\vec{\mathbf{j}} + z_2\vec{\mathbf{k}} - x_1\vec{\mathbf{i}} - y_1\vec{\mathbf{j}} - z_1\vec{\mathbf{k}} =$

$$= (x_2 - x_1)\vec{\mathbf{i}} + (y_2 - y_1)\vec{\mathbf{j}} + (z_2 - z_1)\vec{\mathbf{k}} = \{x_2 - x_1,\ y_2 - y_1,\ z_2 - z_1\}. \qquad \textbf{(2.2)}$$

Remark. As we see, components of the vector are differences of corresponding coordinates of its terminal point and initial point.

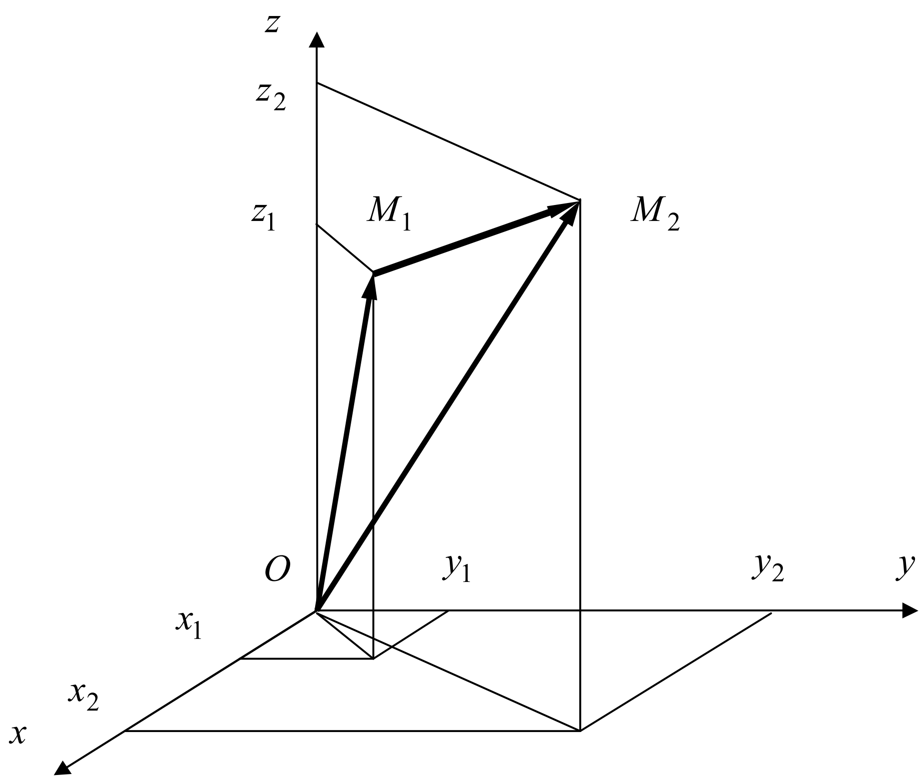


Fig. 2.15

Denote components of $\overrightarrow{\mathbf{M}_1\mathbf{M}_2}$ by $X = x_2 - x_1$, $Y = y_2 - y_1$, $Z = z_2 - z_1$. Then $\overrightarrow{\mathbf{M}_1\mathbf{M}_2} = X\vec{\mathbf{i}} + Y\vec{\mathbf{j}} + Z\vec{\mathbf{k}} = \{X, Y, Z\}$. It is clear that [see Fig. 2.16] line segment M_1M_2 is the diagonal of parallelepiped constructing on the vectors $X\vec{\mathbf{i}}$, $Y\vec{\mathbf{j}}$ and $Z\vec{\mathbf{k}}$. Hence, by the Pythagorean theorem, $\left|\overrightarrow{\mathbf{M}_1\mathbf{M}_2}\right|^2 = |X|^2\left|\vec{\mathbf{i}}\right|^2 + |Y|^2\left|\vec{\mathbf{j}}\right|^2 + |Z|^2\left|\vec{\mathbf{k}}\right|^2 = X^2 + Y^2 + Z^2$.

Therefore the magnitude of $\overrightarrow{\mathbf{M}_1\mathbf{M}_2}$ is

$$\left|\overrightarrow{\mathbf{M}_1\mathbf{M}_2}\right| = \sqrt{X^2 + Y^2 + Z^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}. \quad \textbf{(2.3)}$$

Note that this is the *distance between points* M_1 and M_2.

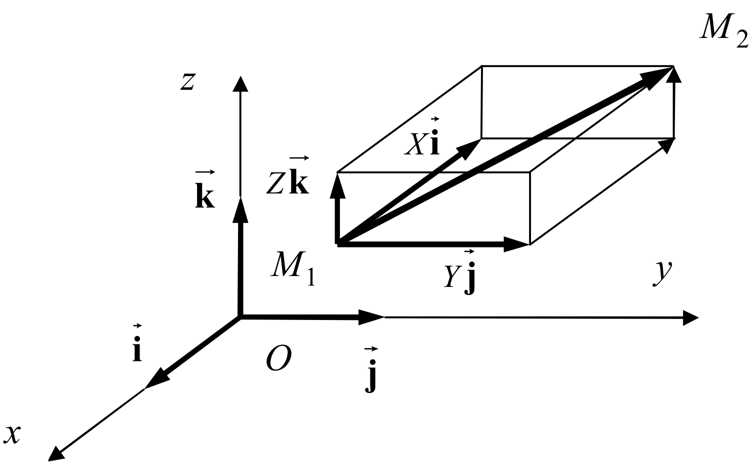


Fig. 2.16

If $\vec{\mathbf{a}} = \{X_a, Y_a, Z_a\}$ and $\vec{\mathbf{b}} = \{X_b, Y_b, Z_b\}$, then the following properties are true.

1. $\vec{\mathbf{a}} = \vec{\mathbf{b}}$ if and only if $X_a = X_b$, $Y_a = Y_b$, $Z_a = Z_b$.

2. $\lambda\vec{\mathbf{a}} = \{\lambda X_a, \lambda Y_a, \lambda Z_a\}$ for λ any scalar. **(2.4)**

3. $\vec{\mathbf{a}} \pm \vec{\mathbf{b}} = \{X_a \pm X_b, Y_a \pm Y_b, Z_a \pm Z_b\}$.

Example 2.2. Find the magnitude of the vector $3\vec{\mathbf{a}}+2\vec{\mathbf{b}}$ if $\vec{\mathbf{a}}=\{-1, 2, -5\}$ and $\vec{\mathbf{b}}$ is the vector joining the point $M_1(3, -2, 0)$ to the point $M_2(4, 2, -3)$.

Solution. The components of the vector $\vec{\mathbf{b}}$ by the formula (2.2) are $X_b = 4-3=1$, $Y_b = 2-(-2)=4$, $Z_b = -3-0=-3$. Using (2.4), we find $3\vec{\mathbf{a}}+2\vec{\mathbf{b}} =\{3\cdot(-1)+2\cdot1,\ 3\cdot2+2\cdot4,\ 3\cdot(-5)+2\cdot(-3)\} = \{-1,\ 14, -21\}$.

Then by the formula (2.3) $\left|3\vec{\mathbf{a}}+2\vec{\mathbf{b}}\right| = \sqrt{(-1)^2+14^2+(-21)^2} = \sqrt{638}\approx 25{,}26.$

2.4. Division of the segment in the preassigned ratio

Let M_1M_2 is the line segment connecting the points $M_1(x_1, y_1, z_1)$ and $M_2(x_2, y_2, z_2)$ [see Fig. 2.17]. Let's find the coordinates of the point $M(x, y, z)$ on the segment, such that M divides the segment in the preassigned ratio $\lambda > 0$, that is, such that $\dfrac{M_1M}{MM_2} = \lambda$.

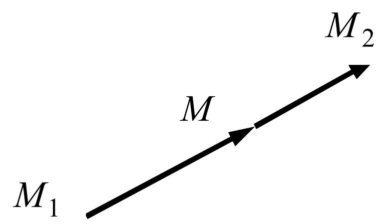


Fig. 2.17

Let's consider the vectors $\overrightarrow{\mathbf{M_1M}}$ and $\overrightarrow{\mathbf{MM_2}}$ [see Fig. 2.17]. Since the direction of the vector $\overrightarrow{\mathbf{M_1M}}$ is the same as to that of $\overrightarrow{\mathbf{MM_2}}$, then $\overrightarrow{\mathbf{M_1M}} = \lambda\,\overrightarrow{\mathbf{MM_2}}$. Taking into account that $\overrightarrow{\mathbf{M_1M}}=\{x-x_1,\ y-y_1,\ z-z_1\}$, $\overrightarrow{\mathbf{MM_2}} = \{x_2-x,\ y_2-y,\ z_2-z\}$, according to $1-2$ of (2.4) we obtain $x-x_1 = \lambda(x_2-x)$, $y-y_1 = \lambda(y_2-y)$, $z-z_1 = \lambda(z_2-z)$. Hence

$$x = \frac{x_1 + \lambda x_2}{1+\lambda}, \quad y = \frac{y_1 + \lambda y_2}{1+\lambda}, \quad z = \frac{z_1 + \lambda z_2}{1+\lambda}. \tag{2.5}$$

In particular, if M is the *midpoint* of the segment, i.e. M *bisects* M_1M_2, then $\frac{M_1M}{MM_2} = 1 = \lambda$ and, from formulas (2.5), we receive

$$x = \frac{x_1 + x_2}{2}, \quad y = \frac{y_1 + y_2}{2}, \quad z = \frac{z_1 + z_2}{2}. \tag{2.6}$$

Thus, the coordinates of the line segment's midpoint are the average values of the respective coordinates of the endpoints.
As application of formulas (2.5) let's consider the problem of searching the centroid's coordinates of a plane region bounded by a triangle with vertices in the points $M_1(x_1, y_1, z_1)$, $M_2(x_2, y_2, z_2)$ and $M_3(x_3, y_3, z_3)$.

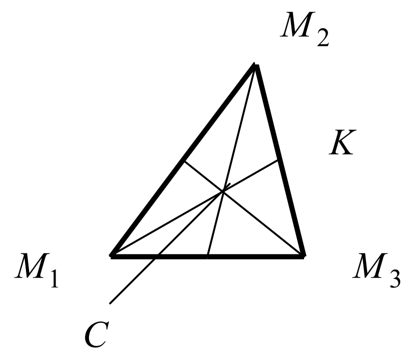


Fig. 2.18

As it is known, required point C is the point of intersection of triangle's medians [see Fig. 2.18]. For example, K is the midpoint of the side M_2M_3. Therefore, as it follows from (2.6), $x_K = \frac{x_2 + x_3}{2}$, $y_K = \frac{y_2 + y_3}{2}$, $z_K = \frac{z_2 + z_3}{2}$. Also it is known, that the point C divides each median in the ratio 2:1 from corresponding vertex. Thus, in particular, $\dfrac{M_1 C}{CK} = \dfrac{2}{1}$. Then from formulas (2.5) by substituting $\lambda = 2$ we finally receive

$$x_C = \frac{x_1 + x_2 + x_3}{3}, \quad y_C = \frac{y_1 + y_2 + y_3}{3}, \quad z_C = \frac{z_1 + z_2 + z_3}{3}. \qquad \textbf{(2.7)}$$

Example 2.3. In the triangle with vertices in the points $A(-3,\ 4,\ 1)$, $B(2,\ -2,\ 3)$ and $C(7,\ -12,\ 3)$ find the coordinates of the point E that bisects the segment AD if it is known that the point D divides the side BC in the ratio 3:2.

Solution. Supposing $\lambda = 3/2$ in formulas (2.5), we'll find the coordinates of the point D : $x_D = \dfrac{x_B + \frac{3}{2}x_C}{1+\frac{3}{2}} = \dfrac{2+\frac{3}{2}\cdot 7}{\frac{5}{2}} = 5$,

$y_D = \dfrac{y_B + \frac{3}{2}y_C}{1+\frac{3}{2}} = \dfrac{-2+\frac{3}{2}\cdot(-12)}{\frac{5}{2}} = -8$, $\quad z_D = \dfrac{z_B + \frac{3}{2}z_C}{1+\frac{3}{2}} = \dfrac{3+\frac{3}{2}\cdot 3}{\frac{5}{2}} = 3$. Then by formulas (2.6) we receive $x_E = \dfrac{x_A + x_D}{2} = \dfrac{-3+5}{2} = 1$, $y_E = \dfrac{y_A + y_D}{2} =$

$= \dfrac{4-8}{2} = -2$, $z_E = \dfrac{z_A + z_D}{2} = \dfrac{1+3}{2} = 2$. Thus $E\,(1,\ -2,\ 2)$.

2.4. Projecting a vector on the axis

Let $\vec{\mathbf{a}}$ is any nonzero vector and $\vec{\mathbf{s}}$ is a certain axis. The ***scalar projection*** of the vector $\vec{\mathbf{a}}$ on the axis $\vec{\mathbf{s}}$, denoted by $\mathbf{Pr}_{\vec{\mathbf{s}}}\,\vec{\mathbf{a}}$, is defined as the product of the magnitude of $\vec{\mathbf{a}}$ and the cosine of the angle between $\vec{\mathbf{a}}$ and $\vec{\mathbf{s}}$. In symbols,

$$\mathbf{Pr}_{\vec{\mathbf{s}}}\,\vec{\mathbf{a}} = \left|\vec{\mathbf{a}}\right| \cos\varphi\,, \qquad \textbf{(2.8)}$$

where $\varphi = (\widehat{\vec{\mathbf{a}}, \vec{\mathbf{s}}})$ $(0 \le \varphi \le \pi)$ is the smaller angle that $\vec{\mathbf{a}}$ makes with the positive $\vec{\mathbf{s}}$-axis [see Fig. 2.19]. It is easy to see that $\mathbf{Pr}_{\vec{\mathbf{s}}}\vec{\mathbf{a}} > 0 \Leftrightarrow 0 \le \varphi < \pi/2$, $\mathbf{Pr}_{\vec{\mathbf{s}}}\vec{\mathbf{a}} = 0 \Leftrightarrow \varphi = \pi/2$ and $\mathbf{Pr}_{\vec{\mathbf{s}}}\vec{\mathbf{a}} < 0 \Leftrightarrow \pi/2 < \varphi \le \pi$, where the symbol "$\Leftrightarrow$", as usually, means " if and only if" or "equivalently".

Remark. It is clear that $\mathbf{Pr}_{\vec{\mathbf{a}}}\vec{\mathbf{a}} = |\vec{\mathbf{a}}|\cos 0 = |\vec{\mathbf{a}}|$. Hence $\mathbf{Pr}_{\vec{\mathbf{i}}}\vec{\mathbf{i}} = \mathbf{Pr}_{\vec{\mathbf{j}}}\vec{\mathbf{j}} = \mathbf{Pr}_{\vec{\mathbf{k}}}\vec{\mathbf{k}} = 1$ because $|\vec{\mathbf{i}}| = |\vec{\mathbf{j}}| = |\vec{\mathbf{k}}| = 1$. Also we note that $\mathbf{Pr}_{\vec{\mathbf{i}}}\vec{\mathbf{j}} = \mathbf{Pr}_{\vec{\mathbf{i}}}\vec{\mathbf{k}} = \mathbf{Pr}_{\vec{\mathbf{j}}}\vec{\mathbf{i}} = \mathbf{Pr}_{\vec{\mathbf{j}}}\vec{\mathbf{k}} =$
$= \mathbf{Pr}_{\vec{\mathbf{k}}}\vec{\mathbf{i}} = \mathbf{Pr}_{\vec{\mathbf{k}}}\vec{\mathbf{j}} = 0$ because $\vec{\mathbf{i}} \perp \vec{\mathbf{j}} \perp \vec{\mathbf{k}}$, i.e. respective angles are equal to $\pi/2$.

If $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ are two nonzero vectors and λ is the scalar, then following properties of the projections are true:

1. $\mathbf{Pr}_{\vec{\mathbf{s}}}(\vec{\mathbf{a}} \pm \vec{\mathbf{b}}) = \mathbf{Pr}_{\vec{\mathbf{s}}}\vec{\mathbf{a}} \pm \mathbf{Pr}_{\vec{\mathbf{s}}}\vec{\mathbf{b}}$ (Chasles's theorem) [see Fig. 2.19].
2. $\mathbf{Pr}_{\vec{\mathbf{s}}}(\lambda\vec{\mathbf{a}}) = \lambda\, \mathbf{Pr}_{\vec{\mathbf{s}}}\vec{\mathbf{a}}$.

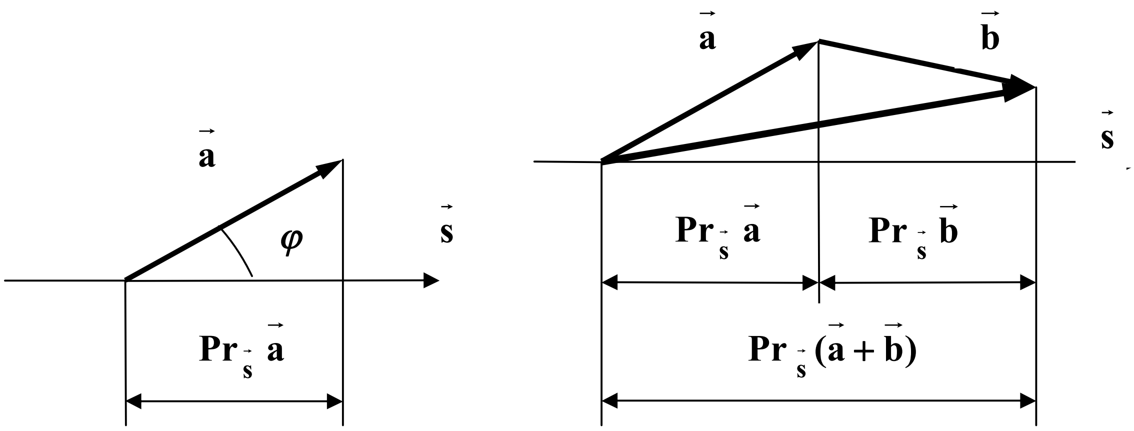


Fig. 2.19

As it follows from (2.8),

$$\cos\varphi = \frac{\mathbf{Pr}_{\vec{\mathbf{s}}}\vec{\mathbf{a}}}{|\vec{\mathbf{a}}|}. \qquad \textbf{(2.9)}$$

Also it is obvious that the projections of the vector $\vec{\mathbf{a}} = \{X_a, Y_a, Z_a\}$ on the positive *Ox*, *Oy* and *Oz* axes are the projections of this vector on respective unit coordinate vectors $\vec{\mathbf{i}}$, $\vec{\mathbf{j}}$, $\vec{\mathbf{k}}$. Taking into account the properties of the projections, and also the stated remark, we receive

$$\mathbf{Pr}_{\mathbf{Ox}}\vec{\mathbf{a}} = \mathbf{Pr}_{\vec{\mathbf{i}}}\vec{\mathbf{a}} = \mathbf{Pr}_{\vec{\mathbf{i}}}\,(X_a\vec{\mathbf{i}} + Y_a\vec{\mathbf{j}} + Z_a\vec{\mathbf{k}}) = X_a\,\mathbf{Pr}_{\vec{\mathbf{i}}}\vec{\mathbf{i}} + Y_a\,\mathbf{Pr}_{\vec{\mathbf{i}}}\vec{\mathbf{j}} + Z_a\,\mathbf{Pr}_{\vec{\mathbf{i}}}\vec{\mathbf{k}} =$$

$= X_a\cdot 1 + Y_a\cdot 0 + Z_a\cdot 0 = X_a$. Likewise, $\mathbf{Pr}_{\mathbf{Oy}}\vec{\mathbf{a}} = Y_a$ and $\mathbf{Pr}_{\mathbf{Oz}}\vec{\mathbf{a}} = Z_a$.

Denoting projections by $a_x = \mathbf{Pr}_{\mathbf{Ox}}\vec{\mathbf{a}}$, $a_y = \mathbf{Pr}_{\mathbf{Oy}}\vec{\mathbf{a}}$, $a_z = \mathbf{Pr}_{\mathbf{Oz}}\vec{\mathbf{a}}$, we finally obtain

$$a_x = X_a,\ a_y = Y_a,\ a_z = Z_a, \qquad \textbf{(2.11)}$$

i.e. *the scalar projections of a vector on the coordinate axes are the same as respective components of this vector.*

2.5. Direction cosines of a vector

Let the vector $\vec{\mathbf{a}} = X_a\vec{\mathbf{i}} + Y_a\vec{\mathbf{j}} + Z_a\vec{\mathbf{k}}$ make angles α, β and γ, respectively, with the positive *Ox*, *Oy* and *Oz* axes, as in Fig. 2.20.

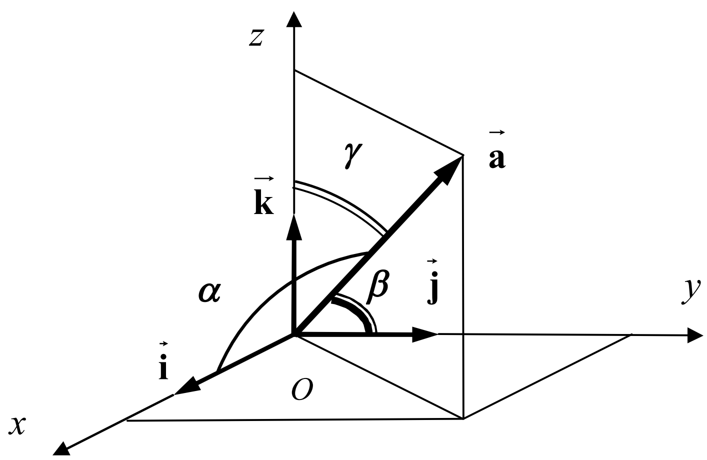


Fig. 2.20

Then, taking into account, that $\alpha = (\vec{\mathbf{a}}\,\widehat{,}\,\vec{\mathbf{i}})$, $\beta = (\vec{\mathbf{a}}\,\widehat{,}\,\vec{\mathbf{j}})$ and $\gamma = (\vec{\mathbf{a}}\,\widehat{,}\,\vec{\mathbf{k}})$, and by formulas (2.3), (2.9), (2.11), we obtain so-called the ***direction cosines*** of $\vec{\mathbf{a}}$

$$\cos\alpha = \frac{\mathbf{Pr}_{\mathbf{Ox}}\vec{\mathbf{a}}}{|\vec{\mathbf{a}}|} = \frac{\mathbf{Pr}_{\vec{\mathbf{i}}}\vec{\mathbf{a}}}{|\vec{\mathbf{a}}|} = \frac{X_a}{\sqrt{X_a^2 + Y_a^2 + Z_a^2}},$$

$$\cos\beta = \frac{\mathbf{Pr}_{\mathbf{Oy}}\vec{\mathbf{a}}}{|\vec{\mathbf{a}}|} = \frac{\mathbf{Pr}_{\vec{\mathbf{j}}}\vec{\mathbf{a}}}{|\vec{\mathbf{a}}|} = \frac{Y_a}{\sqrt{X_a^2 + Y_a^2 + Z_a^2}}, \qquad \mathbf{(2.12)}$$

$$\cos\gamma = \frac{\mathbf{Pr}_{\mathbf{Oz}}\vec{\mathbf{a}}}{|\vec{\mathbf{a}}|} = \frac{\mathbf{Pr}_{\vec{\mathbf{k}}}\vec{\mathbf{a}}}{|\vec{\mathbf{a}}|} = \frac{Z_a}{\sqrt{X_a^2 + Y_a^2 + Z_a^2}}.$$

Since, as it easy to see,

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1, \qquad \mathbf{(2.13)}$$

the vector $\cos\alpha \cdot \vec{\mathbf{i}} + \cos\beta \cdot \vec{\mathbf{j}} + \cos\gamma \cdot \vec{\mathbf{k}}$ is the unit vector in the direction of $\vec{\mathbf{a}}$, i.e.

$$\overrightarrow{\mathbf{a}^\circ} = \frac{\vec{\mathbf{a}}}{|\vec{\mathbf{a}}|} = \{\cos\alpha, \cos\beta, \cos\gamma\}. \qquad \mathbf{(2.14)}$$

Note that (2.13) is an important property of the direction cosines.

<u>**Example 2.4.**</u> Find the component Y of the vector $\vec{\mathbf{a}} = \{9,\ Y,\ \sqrt{59}\}$ if $\cos\alpha = \sqrt{2}/3$ and $\cos\beta = -\sqrt{3}/2$.

Solution. Using the property of the direction cosines (2.13), we obtain $\cos^2\gamma = 1 - \cos^2\alpha - \cos^2\beta = 1 - \frac{2}{9} - \frac{3}{4} = \frac{1}{36}$. Then from the last formula (2.12) it follows that $\frac{Y^2}{9^2 + Y^2 + (\sqrt{59})^2} = \frac{1}{36}$, whence $35Y^2 = 140$. Thus $Y = \pm\, 2$.

2. Scalar product

2.7. Concept of a scalar (or dot) product

The ***scalar product*** of vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$, denoted by $\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}$ (read $\vec{\mathbf{a}}$ dot $\vec{\mathbf{b}}$) is defined as the product of the magnitudes of $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ and the cosine of the angle between them. In symbols,

$$\vec{\mathbf{a}}\cdot\vec{\mathbf{b}} = \left|\vec{\mathbf{a}}\right|\left|\vec{\mathbf{b}}\right|\cos\varphi \qquad \mathbf{(2.15)}$$

where $\varphi = (\vec{\mathbf{a}},\widehat{\ }\vec{\mathbf{b}})$ $(0\le\varphi\le\pi)$ is the smaller angle between the two vectors when they are drawn with a common initial point [see 2.1 and Fig. 2.3]. The scalar product is frequently also called the ***dot product***. Note that $\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}$ ***is a scalar and not a vector.***

2.8. Fundamental properties of a scalar product

From the definition we can derive the following properties of the scalar product.

1. $\vec{\mathbf{a}}\cdot\vec{\mathbf{b}} = \vec{\mathbf{b}}\cdot\vec{\mathbf{a}}$ (*Commutative Law*).

2. $\lambda\cdot(\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}) = (\lambda\,\vec{\mathbf{a}})\cdot\vec{\mathbf{b}} = \vec{\mathbf{a}}\cdot(\lambda\vec{\mathbf{b}})$

and $(\lambda\cdot\vec{\mathbf{a}})\cdot(\mu\cdot\vec{\mathbf{b}}) = (\lambda\cdot\mu)\cdot(\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}) = (\mu\cdot\vec{\mathbf{a}})\cdot(\lambda\vec{\mathbf{b}})$ where λ and μ are the scalars (*Associative Law for Scalar Multiplier*).

3. $\vec{\mathbf{a}}\cdot(\vec{\mathbf{b}}+\vec{\mathbf{c}}) = \vec{\mathbf{a}}\cdot\vec{\mathbf{b}}+\vec{\mathbf{a}}\cdot\vec{\mathbf{c}}$ (*Distributive Law*).

4. $\vec{\mathbf{a}}\cdot\vec{\mathbf{a}} = \left|\vec{\mathbf{a}}\right|\left|\vec{\mathbf{a}}\right|\cos 0 = \left|\vec{\mathbf{a}}\right|^2$, whence $\left|\vec{\mathbf{a}}\right| = \sqrt{\vec{\mathbf{a}}\cdot\vec{\mathbf{a}}}$.

In particular, $\vec{\mathbf{i}}\cdot\vec{\mathbf{i}} = \vec{\mathbf{j}}\cdot\vec{\mathbf{j}} = \vec{\mathbf{k}}\cdot\vec{\mathbf{k}} = 1$ because $\left|\vec{\mathbf{i}}\right| = \left|\vec{\mathbf{j}}\right| = \left|\vec{\mathbf{k}}\right| = 1$ and

$\vec{\mathbf{i}} \cdot \vec{\mathbf{j}} = \vec{\mathbf{j}} \cdot \vec{\mathbf{k}} = \vec{\mathbf{k}} \cdot \vec{\mathbf{i}} = 0$ because $\vec{\mathbf{i}} \perp \vec{\mathbf{j}} \perp \vec{\mathbf{k}}$.

5. $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = 0 \Leftrightarrow \{\vec{\mathbf{a}} = \vec{\mathbf{0}}$ or $\vec{\mathbf{b}} = \vec{\mathbf{0}}$ or $\vec{\mathbf{a}} = \vec{\mathbf{b}} = \vec{\mathbf{0}}$ or $\vec{\mathbf{a}} \perp \vec{\mathbf{b}}\}$.

Remark. The symbol "$\perp$" usually means that vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ are ***orthogonal*** (or perpendicular).The terms "perpendicular" and "orthogonal" almost mean the same thing. Perpendicularity of vectors means nothing but $\varphi = \pi/2$. But the zero vector has no direction, so technically speaking, the zero vector is not perpendicular to any vector. However, the zero vector is orthogonal to every vector. Except to this special case, orthogonal and perpendicular are the same. Then we say $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = 0 \Leftrightarrow \vec{\mathbf{a}} \perp \vec{\mathbf{b}}$ (i.e. $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = 0$ if and only if $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ are *perpendicular*).

Also we note that $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} > 0 \Leftrightarrow 0 \le \varphi < \pi/2$ and $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} < 0 \Leftrightarrow \pi/2 < \varphi \le \pi$.

2.9. Some applications of a scalar product

2.9.1 Angle between vectors

We can use the scalar product to find the ***angle*** $\boldsymbol{\varphi}$ between any two nonzero vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$.

As it follows from (2.8),

$$\cos\varphi = \frac{\vec{\mathbf{a}} \cdot \vec{\mathbf{b}}}{|\vec{\mathbf{a}}| \cdot |\vec{\mathbf{b}}|}. \tag{2.16}$$

Hence

$$\varphi = \arccos\left(\frac{\vec{\mathbf{a}} \cdot \vec{\mathbf{b}}}{|\vec{\mathbf{a}}| \cdot |\vec{\mathbf{b}}|}\right). \tag{2.17}$$

2.9.2 Projecting one vector onto another

The ***scalar projection*** of any vector $\vec{\mathbf{a}}$ on any nonzero vector $\vec{\mathbf{b}}$, denoted by $\mathbf{Pr}_{\vec{\mathbf{b}}}\vec{\mathbf{a}}$, in accord with (2.8), is defined as the product of the magnitude of $\vec{\mathbf{a}}$ and the cosine of the angle φ between $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ [see Fig. 2.21]. In symbols,

$$\mathbf{Pr}_{\vec{\mathbf{b}}}\vec{\mathbf{a}} = \left|\vec{\mathbf{a}}\right|\mathbf{cos}\varphi . \tag{2.18}$$

Substituting instead of $\mathbf{cos}\,\varphi$ its expression (2.9) we obtain

$$\mathbf{Pr}_{\vec{\mathbf{b}}}\vec{\mathbf{a}} = \left|\vec{\mathbf{a}}\right| \cdot \frac{\vec{\mathbf{a}} \cdot \vec{\mathbf{b}}}{\left|\vec{\mathbf{a}}\right| \cdot \left|\vec{\mathbf{b}}\right|} = \frac{\vec{\mathbf{a}} \cdot \vec{\mathbf{b}}}{\left|\vec{\mathbf{b}}\right|}. \tag{2.19}$$

Likewise, the scalar projection of $\vec{\mathbf{b}}$ on $\vec{\mathbf{a}}$ is

$$\mathbf{Pr}_{\vec{\mathbf{a}}}\vec{\mathbf{b}} = \left|\vec{\mathbf{b}}\right| \cdot \frac{\vec{\mathbf{a}} \cdot \vec{\mathbf{b}}}{\left|\vec{\mathbf{a}}\right| \cdot \left|\vec{\mathbf{b}}\right|} = \frac{\vec{\mathbf{a}} \cdot \vec{\mathbf{b}}}{\left|\vec{\mathbf{a}}\right|} \tag{2.20}$$

[see Fig. 2.21].

<u>Note.</u> $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}}$ is the product of the length of $\vec{\mathbf{a}}$ and the scalar projection of $\vec{\mathbf{b}}$ on $\vec{\mathbf{a}}$, i.e. $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = \left|\vec{\mathbf{a}}\right| \cdot \mathbf{Pr}_{\vec{\mathbf{a}}}\vec{\mathbf{b}}$. Likewise, $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = \left|\vec{\mathbf{b}}\right| \cdot \mathbf{Pr}_{\vec{\mathbf{b}}}\vec{\mathbf{a}}$.

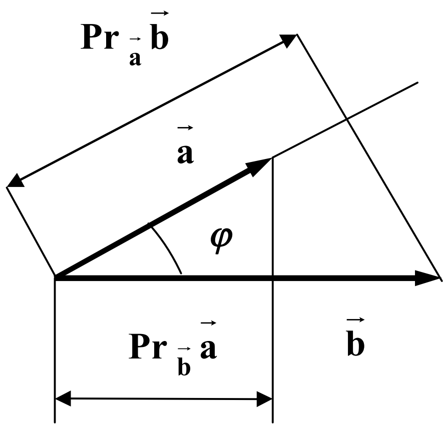


Fig. 2.21

The ***vector projection*** of any vector $\vec{\mathbf{a}}$ on any nonzero vector $\vec{\mathbf{b}}$, denoted by $\overrightarrow{\mathbf{Pr}}_{\vec{\mathbf{b}}}\vec{\mathbf{a}}$, is defined as the product of the scalar projection of $\vec{\mathbf{a}}$ on $\vec{\mathbf{b}}$ and the unit vector in the direction of $\vec{\mathbf{b}}$. In symbols,

$$\overrightarrow{\mathbf{Pr}}_{\vec{\mathbf{b}}}\vec{\mathbf{a}} = \mathbf{Pr}_{\vec{\mathbf{b}}}\vec{\mathbf{a}}\cdot\overrightarrow{\mathbf{b}^{\circ}} = \left(\vec{\mathbf{a}}\cdot\frac{\vec{\mathbf{b}}}{\left|\vec{\mathbf{b}}\right|}\right)\cdot\frac{\vec{\mathbf{b}}}{\left|\vec{\mathbf{b}}\right|} = \left(\frac{\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}}{\left|\vec{\mathbf{b}}\right|^{2}}\right)\cdot\vec{\mathbf{b}}. \qquad \textbf{(2.21)}$$

Likewise, the vector projection of $\vec{\mathbf{b}}$ on $\vec{\mathbf{a}}$ is

$$\overrightarrow{\mathbf{Pr}}_{\vec{\mathbf{a}}}\vec{\mathbf{b}} = \mathbf{Pr}_{\vec{\mathbf{a}}}\vec{\mathbf{b}}\cdot\overrightarrow{\mathbf{a}^{\circ}} = \left(\vec{\mathbf{b}}\cdot\frac{\vec{\mathbf{a}}}{\left|\vec{\mathbf{a}}\right|}\right)\cdot\frac{\vec{\mathbf{a}}}{\left|\vec{\mathbf{a}}\right|} = \left(\frac{\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}}{\left|\vec{\mathbf{a}}\right|^{2}}\right)\cdot\vec{\mathbf{a}}. \qquad \textbf{(2.22)}$$

Note. If $\vec{\mathbf{a}}$ is a force vector, then $\overrightarrow{\mathbf{Pr}}_{\vec{\mathbf{b}}}\vec{\mathbf{a}}$ represents the ***effective force*** in the direction of $\vec{\mathbf{b}}$ [see Fig. 2.22].

We can use vector projections to determine the amount of force required in different problems.

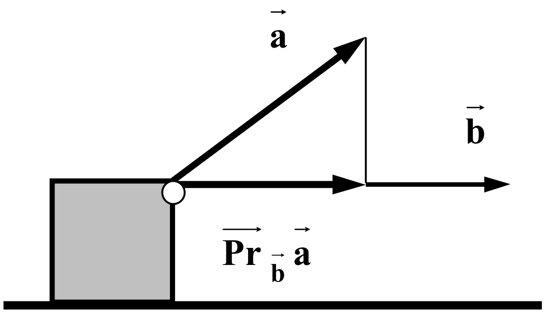


Fig. 2.22

2.9.3 Work

If $\vec{\mathbf{F}}$ is a constant force, then the ***work*** $\mathbf{W}$ done by $\vec{\mathbf{F}}$ in moving an object from the initial point of $\vec{\mathbf{s}}$ to its terminal point is

$$\mathbf{W} = \vec{\mathbf{F}}\cdot\vec{\mathbf{s}} = \left|\vec{\mathbf{F}}\right|\left|\vec{\mathbf{s}}\right|\cos\varphi, \qquad \textbf{(2.23)}$$

where $\varphi = (\widehat{\vec{\mathbf{F}}, \vec{\mathbf{s}}})$.

Remark. From (2.21) it follows that the work is a scalar product of the effective force in the direction of $\vec{\mathbf{s}}$ with $\vec{\mathbf{s}}$, i.e. $\mathbf{W} = \overrightarrow{\mathbf{Pr}_{\vec{\mathbf{s}}}\,\mathbf{F}} \cdot \vec{\mathbf{s}}$.

2.10. Expressing of the scalar product and its applications in terms of rectangular coordinates

Let's find directly the scalar product of the vectors $\vec{\mathbf{a}} = \{X_a, Y_a, Z_a\}$ and $\vec{\mathbf{b}} = \{X_b, Y_b, Z_b\}$:

$$\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = (X_a \vec{\mathbf{i}} + Y_a \vec{\mathbf{j}} + Z_a \vec{\mathbf{k}}) \cdot (X_b \vec{\mathbf{i}} + Y_b \vec{\mathbf{j}} + Z_b \vec{\mathbf{k}}) =$$

$$= X_a X_b \vec{\mathbf{i}} \cdot \vec{\mathbf{i}} + X_a Y_b \vec{\mathbf{i}} \cdot \vec{\mathbf{j}} + X_a Z_b \vec{\mathbf{i}} \cdot \vec{\mathbf{k}} + Y_a X_b \vec{\mathbf{j}} \cdot \vec{\mathbf{i}} + Y_a Y_b \vec{\mathbf{j}} \cdot \vec{\mathbf{j}} +$$

$$+ Y_a Z_b \vec{\mathbf{j}} \cdot \vec{\mathbf{k}} + Z_a X_b \vec{\mathbf{k}} \cdot \vec{\mathbf{i}} + Z_a Y_b \vec{\mathbf{k}} \cdot \vec{\mathbf{j}} + Z_a Z_b \vec{\mathbf{k}} \cdot \vec{\mathbf{k}}.$$

According to property 4 of a scalar product finally we receive

$$\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = X_a X_b + Y_a Y_b + Z_a Z_b. \qquad \textbf{(2.24)}$$

Thus we see that is remarkably simple to compute the scalar product of two vectors when we know their components.

The condition of perpendicularity of two nonzero vectors now takes a form

$$X_a X_b + Y_a Y_b + Z_a Z_b = 0 \Leftrightarrow \vec{\mathbf{a}} \perp \vec{\mathbf{b}}. \qquad \textbf{(2.25)}$$

Substituting (2.24) into (2.16), (2.19) – (2.22), we obtain

- the cosine of the angle between $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$

$$\mathbf{cos}\,\varphi = \frac{X_a X_b + Y_a Y_b + Z_a Z_b}{\sqrt{X_a^2 + Y_a^2 + Z_a^2}\,\sqrt{X_b^2 + Y_b^2 + Z_b^2}}. \qquad \textbf{(2.26)}$$

Note. If $\cos\alpha_1$, $\cos\beta_1$ and $\cos\gamma_1$ are the direction cosines of $\vec{\mathbf{a}}$ and $\cos\alpha_2$, $\cos\beta_2$ and $\cos\gamma_2$ are the direction cosines of $\vec{\mathbf{b}}$, then, as it follows from (2.12) and (2.26),

$$\cos\varphi = \cos\alpha_1 \cos\alpha_2 + \cos\beta_1 \cos\beta_2 + \cos\gamma_1 \cos\gamma_2. \qquad \textbf{(2.27)}$$

- the scalar projections

$$\mathbf{Pr}_{\vec{\mathbf{b}}}\vec{\mathbf{a}} = \frac{\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}}{|\vec{\mathbf{b}}|} = \frac{X_a X_b + Y_a Y_b + Z_a Z_b}{\sqrt{X_b^2 + Y_b^2 + Z_b^2}}, \qquad \mathbf{(2.28)}$$

$$\mathbf{Pr}_{\vec{\mathbf{a}}}\vec{\mathbf{b}} = \frac{\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}}{|\vec{\mathbf{a}}|} = \frac{X_a X_b + Y_a Y_b + Z_a Z_b}{\sqrt{X_a^2 + Y_a^2 + Z_a^2}}. \qquad \mathbf{(2.29)}$$

Note. With use of (2.24) the correlations (2.11) may be proved more simply, namely

$$a_x = \mathbf{Pr}_{\vec{\mathbf{i}}}\vec{\mathbf{a}} = \frac{\vec{\mathbf{a}}\cdot\vec{\mathbf{i}}}{|\vec{\mathbf{i}}|} = X_a,\ a_y = \mathbf{Pr}_{\vec{\mathbf{j}}}\vec{\mathbf{a}} = \frac{\vec{\mathbf{a}}\cdot\vec{\mathbf{j}}}{|\vec{\mathbf{j}}|} = Y_a,\ a_z = \mathbf{Pr}_{\vec{\mathbf{k}}}\vec{\mathbf{a}} = \frac{\vec{\mathbf{a}}\cdot\vec{\mathbf{k}}}{|\vec{\mathbf{k}}|} = Z_a,$$

where $\vec{\mathbf{a}} = \{X_a, Y_a, Z_a\}$, $\vec{\mathbf{i}} = \{1, 0, 0\}$, $\vec{\mathbf{j}} = \{0, 1, 0\}$, $\vec{\mathbf{k}} = \{0, 0, 1\}$, $|\vec{\mathbf{i}}| = |\vec{\mathbf{j}}| = |\vec{\mathbf{k}}| = 1$.

- the vector projections

$$\overrightarrow{\mathbf{Pr}}_{\vec{\mathbf{b}}}\vec{\mathbf{a}} = \mathrm{B}\,(X_b\,\vec{\mathbf{i}} + Y_b\,\vec{\mathbf{j}} + Z_b\,\vec{\mathbf{k}}), \qquad \mathbf{(2.30)}$$

$$\overrightarrow{\mathbf{Pr}}_{\vec{\mathbf{a}}}\vec{\mathbf{b}} = \mathrm{A}\,(X_a\,\vec{\mathbf{i}} + Y_a\,\vec{\mathbf{j}} + Z_a\,\vec{\mathbf{k}}), \qquad \mathbf{(2.31)}$$

where $\mathrm{B} = \dfrac{X_a X_b + Y_a Y_b + Z_a Z_b}{X_b^2 + Y_b^2 + Z_b^2}$, $\mathrm{A} = \dfrac{X_a X_b + Y_a Y_b + Z_a Z_b}{X_a^2 + Y_a^2 + Z_a^2}$.

- the work done by the force $\vec{\mathbf{F}} = \{X_F, Y_F, Z_F\}$ on $\vec{\mathbf{s}} = \overrightarrow{\mathbf{AB}}$ from the point $A(x_A, y_A, z_A)$ to the point $B(x_B, y_B, z_B)$

$$\mathbf{W} = X_F(x_B - x_A) + Y_F(y_B - y_A) + Z_F(z_B - z_A). \qquad \mathbf{(2.32)}$$

<u>Example 2.5.</u> Find the angle between vectors $2\vec{\mathbf{a}} - 3\vec{\mathbf{b}}$ and $\vec{\mathbf{a}} + 2\vec{\mathbf{b}}$, and also $\mathbf{Pr}_{\vec{\mathbf{a}}+2\vec{\mathbf{b}}}(2\vec{\mathbf{a}} - 3\vec{\mathbf{b}})$, if $\vec{\mathbf{a}} = 3\vec{\mathbf{i}} - 2\vec{\mathbf{j}} + 4\vec{\mathbf{k}}$ and $\vec{\mathbf{b}} = \{-1, 2, -3\}$.

Solution. Let's find the coordinates of required vectors, using (2.4): $2\vec{\mathbf{a}} - 3\vec{\mathbf{b}} = \{2\cdot 3 - 3\cdot(-1),\ 2\cdot(-2) - 3\cdot 2,\ 2\cdot 4 - 3\cdot(-3)\} = \{9, -10, 17\}$, $\vec{\mathbf{a}} + 2\vec{\mathbf{b}} = \{3 + 2\cdot(-1),\ -2 + 2\cdot 2,\ 4 + 2\cdot(-3)\} = \{1, 2, -2\}$. Then by formula (2.24) we receive $(2\vec{\mathbf{a}} - 3\vec{\mathbf{b}})\cdot(\vec{\mathbf{a}} + 2\vec{\mathbf{b}}) = 9\cdot 1 + (-10)\cdot 2 + 17\cdot(-2) = -45$. Since

$\left|2\vec{\mathbf{a}}-3\vec{\mathbf{b}}\right|=\sqrt{9^2+(-10)^2+17^2}=\sqrt{470}$, $\left|\vec{\mathbf{a}}+2\vec{\mathbf{b}}\right|=\sqrt{1^2+2^2+(-2)^2}=3$, then by formula (2.16) we find $\cos\varphi=\dfrac{(2\vec{\mathbf{a}}-3\vec{\mathbf{b}})\cdot(\vec{\mathbf{a}}+2\vec{\mathbf{b}})}{\left|2\vec{\mathbf{a}}-3\vec{\mathbf{b}}\right|\left|\vec{\mathbf{a}}+2\vec{\mathbf{b}}\right|}=-\dfrac{45}{\sqrt{470}\cdot 3}\approx-0{,}692$,

whence $\varphi\cong\pi-\mathbf{arccos}\ 0{,}692=2{,}34$ [rad.] (or $\varphi\cong 133^\circ 47'$). Now by formula (2.19) we calculate $\mathbf{Pr}_{\vec{a}+2\vec{b}}(2\vec{\mathbf{a}}-3\vec{\mathbf{b}})=\dfrac{(2\vec{\mathbf{a}}-3\vec{\mathbf{b}})\cdot(\vec{\mathbf{a}}+2\vec{\mathbf{b}})}{\left|\vec{\mathbf{a}}+2\vec{\mathbf{b}}\right|}=-\dfrac{45}{3}=-15.$

Example 2.6. Find the effective force $\overrightarrow{\mathbf{F_e}}$ in the direction of the vector $\vec{\mathbf{d}}=3\vec{\mathbf{i}}+5\vec{\mathbf{j}}-4\vec{\mathbf{k}}$ and a magnitude of this force if $\vec{\mathbf{F}}=\{7,7,-11\}$.

Solution. Using the formula (2.30) where $\vec{\mathbf{a}}=\vec{\mathbf{F}}$ and $\vec{\mathbf{b}}=\vec{\mathbf{d}}$, we obtain the effective force

$$\overrightarrow{\mathbf{F_e}}=\frac{7\cdot 3+7\cdot 5+(-11)\cdot(-4)}{3^2+5^2+(-4)^2}\{3,5,-4\}=\frac{100}{50}\{3,5,-4\}=6\vec{\mathbf{i}}+10\vec{\mathbf{j}}-8\vec{\mathbf{k}}.$$

Hence the magnitude of the effective force is

$$\left|\overrightarrow{\mathbf{F_e}}\right|=\sqrt{6^2+10^2+(-8)^2}=10\sqrt{2}\approx 14{,}14.$$

Example 2.7. Find the work done by a 30 Newton force acting in the direction $\vec{\mathbf{d}}=\{-1,2,-2\}$ in moving an object from $A(3, 1, 0)$ to $B(6, -3, 12)$.

Solution. The force $\vec{\mathbf{F}}$ has magnitude $\left|\vec{\mathbf{F}}\right|=30$ and acts in the direction $\vec{\mathbf{d}}$, so

$$\vec{\mathbf{F}}=\left|\vec{\mathbf{F}}\right|\frac{\vec{\mathbf{d}}}{\left|\vec{\mathbf{d}}\right|}=\left|\vec{\mathbf{F}}\right|\overrightarrow{\mathbf{d}^\circ}=\frac{30}{\sqrt{(-1)^2+2^2+(-2)^2}}(-\vec{\mathbf{i}}+2\vec{\mathbf{j}}-2\vec{\mathbf{k}})=10(-\vec{\mathbf{i}}+2\vec{\mathbf{j}}-2\vec{\mathbf{k}}).$$

The direction of motion is from $A(3, 1, 0)$ to $B(6, -3, 12)$, so $\overrightarrow{\mathbf{AB}}=\{3,-4,12\}$.

Thus [see formula (2.32)], the work done by the force is

$$\mathbf{W}=\vec{\mathbf{F}}\cdot\overrightarrow{\mathbf{AB}}=10\big[(-1)\cdot 3+2\cdot(-4)+(-2)\cdot 12\big]=-350\,[\text{Joules}].$$

3. Vector and triple scalar products

In mathematics, the ***vector product*** is a *binary operation* on vectors in a three-dimensional space. It differs from the scalar product in that it results in a vector rather than in a scalar. The idea of vector product is motivated by the wish to find a vector that is perpendicular to two given vectors. The vector product and the scalar product are two ways of multiplying vectors which see the wide applications in geometry, physics and engineering.

2.11. Concept of a vector product and its geometric meaning

Let $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ be two *noncollinear* vectors. Their ***vector*** (or ***cross***, or ***outer***, or ***external***) ***product*** is defined as a vector $\vec{\mathbf{a}}\times\vec{\mathbf{b}}$ perpendicular to both $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ and whose

- magnitude is the product of the magnitude of $\vec{\mathbf{a}}$, the magnitude of $\vec{\mathbf{b}}$, and the sine of the smaller angle φ ($0\le\varphi\le\pi$) between $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$, i.e.

$$\left|\vec{\mathbf{a}}\times\vec{\mathbf{b}}\right| = \left|\vec{\mathbf{a}}\right|\left|\vec{\mathbf{b}}\right|\mathbf{sin}\,\varphi; \qquad \mathbf{(2.33)}$$

- direction is such that the three vectors $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$ and $\vec{\mathbf{a}}\times\vec{\mathbf{b}}$ (in that order) form a right-handed triad.

Thus,

$$\vec{\mathbf{a}}\times\vec{\mathbf{b}} = \left|\vec{\mathbf{a}}\right|\left|\vec{\mathbf{b}}\right|\mathbf{sin}\,\varphi\ \overrightarrow{\mathbf{n}^{\circ}}, \qquad \mathbf{(2.34)}$$

where $\overrightarrow{\mathbf{n}^{\circ}}$ is the unit vector normal to the plane of $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$, so directed that $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$ and $\overrightarrow{\mathbf{n}^{\circ}}$ form a right-handed triad as in Fig. 2.23.

The magnitude of a vector product $\left|\vec{\mathbf{a}}\times\vec{\mathbf{b}}\right| = \left|\vec{\mathbf{a}}\right|\left|\vec{\mathbf{b}}\right|\mathbf{sin}\,\varphi$ can be interpreted as the unsigned area of the parallelogram whose nonparallel sides are $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$:

$\mathbf{S} = \left|\vec{\mathbf{a}}\right|\mathbf{h}$ ($\mathbf{h} = \left|\vec{\mathbf{b}}\right|\mathbf{sin}\,\varphi$) [see Fig. 2.23].

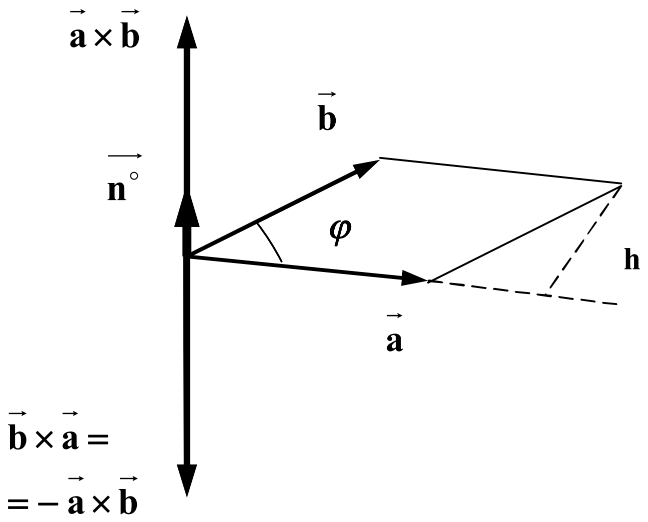


Fig. 2.23

If we observe that $\left|\vec{\mathbf{a}} \times \vec{\mathbf{b}}\right| = \left|\vec{\mathbf{a}}\right|\left|\vec{\mathbf{b}}\right|\sqrt{1-\mathbf{cos}^2\varphi}$, then, using (2.16), we receive very useful formula

$$\left|\vec{\mathbf{a}} \times \vec{\mathbf{b}}\right| = \sqrt{\left|\vec{\mathbf{a}}\right|^2\left|\vec{\mathbf{b}}\right|^2 - (\vec{\mathbf{a}} \cdot \vec{\mathbf{b}})^2}\,. \qquad \mathbf{(2.35)}$$

2.12. Algebraic properties of the vector product

1. If the order of $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ is reversed, then in (2.34) $\overrightarrow{\mathbf{n}^{\circ}}$ must be replaced by $-\overrightarrow{\mathbf{n}^{\circ}}$. Hence,

$$\vec{\mathbf{a}} \times \vec{\mathbf{b}} = -\vec{\mathbf{b}} \times \vec{\mathbf{a}}\,, \qquad \mathbf{(2.36)}$$

i.e. the vector product is *anticommutative*.

Since the unit coordinate vectors $\vec{\mathbf{i}}$, $\vec{\mathbf{j}}$, $\vec{\mathbf{k}}$ form right-handed triad, it follows that

$$\vec{\mathbf{i}} \times \vec{\mathbf{j}} = \vec{\mathbf{k}}, \qquad \vec{\mathbf{j}} \times \vec{\mathbf{k}} = \vec{\mathbf{i}}, \qquad \vec{\mathbf{k}} \times \vec{\mathbf{i}} = \vec{\mathbf{j}}, \qquad \mathbf{(2.37)}$$

$$\vec{\mathbf{j}} \times \vec{\mathbf{i}} = -\vec{\mathbf{k}}, \qquad \vec{\mathbf{k}} \times \vec{\mathbf{j}} = -\vec{\mathbf{i}}, \qquad \vec{\mathbf{i}} \times \vec{\mathbf{k}} = -\vec{\mathbf{j}}\,.$$

2. $\vec{\mathbf{a}} \times (\vec{\mathbf{b}} + \vec{\mathbf{c}}) = \vec{\mathbf{a}} \times \vec{\mathbf{b}} + \vec{\mathbf{a}} \times \vec{\mathbf{c}}$ (*Distributive Law*). **(2.38)**

Notes. **1.** Also $(\vec{\mathbf{a}} + \vec{\mathbf{b}}) \times (\vec{\mathbf{c}} + \vec{\mathbf{d}}) = \vec{\mathbf{a}} \times \vec{\mathbf{c}} + \vec{\mathbf{a}} \times \vec{\mathbf{d}} + \vec{\mathbf{b}} \times \vec{\mathbf{c}} + \vec{\mathbf{b}} \times \vec{\mathbf{d}}$.

2. If $\vec{\mathbf{a}}\times\vec{\mathbf{b}} = \vec{\mathbf{a}}\times\vec{\mathbf{c}}$ and $\vec{\mathbf{a}}\neq\vec{\mathbf{0}}$ then $\vec{\mathbf{a}}\times\vec{\mathbf{b}}-\vec{\mathbf{a}}\times\vec{\mathbf{c}}=\vec{\mathbf{0}}$ and, by the distributive law above $\vec{\mathbf{a}}\times(\vec{\mathbf{b}}-\vec{\mathbf{c}})=\vec{\mathbf{0}}$. Now, if $\vec{\mathbf{a}}$ is parallel to $(\vec{\mathbf{b}}-\vec{\mathbf{c}})$, then even if $\vec{\mathbf{a}}\neq\vec{\mathbf{0}}$ it is possible that $(\vec{\mathbf{b}}-\vec{\mathbf{c}})\neq\vec{\mathbf{0}}$ and therefore that $\vec{\mathbf{b}}\neq\vec{\mathbf{c}}$.
However, if both $\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}=\vec{\mathbf{a}}\cdot\vec{\mathbf{c}}$ *and* $\vec{\mathbf{a}}\times\vec{\mathbf{b}}=\vec{\mathbf{a}}\times\vec{\mathbf{c}}$, then we *can* conclude that $\vec{\mathbf{b}}=\vec{\mathbf{c}}$. This is because if $(\vec{\mathbf{b}}-\vec{\mathbf{c}})\neq\vec{\mathbf{0}}$, then it obviously cannot be *both parallel and perpendicular* to another nonzero vector $\vec{\mathbf{a}}$.

3. Vector product is compatible with a scalar multiplication so that

$$\lambda(\vec{\mathbf{a}}\times\vec{\mathbf{b}})=(\lambda\vec{\mathbf{a}})\times\vec{\mathbf{b}}=\vec{\mathbf{a}}\times(\lambda\vec{\mathbf{b}})=(\vec{\mathbf{a}}\times\vec{\mathbf{b}})\lambda, \qquad \textbf{(2.39)}$$

where λ is a scalar.

4. If two nonzero vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ are parallel or $\vec{\mathbf{a}}=\vec{\mathbf{b}}$, then $\varphi = 0$ (or $\varphi = \pi$) and $\sin\varphi=0$, hence $\vec{\mathbf{a}}\times\vec{\mathbf{b}}=\vec{\mathbf{0}}$. On the contrary, if $\vec{\mathbf{a}}\times\vec{\mathbf{b}}=\vec{\mathbf{0}}$ and $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ are nonzero vectors, then $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ are parallel (we'll denote this fact by $\vec{\mathbf{a}}\,|\,|\,\vec{\mathbf{b}}$). Thus,

$$\vec{\mathbf{a}}\times\vec{\mathbf{b}}=\vec{\mathbf{0}} \Leftrightarrow \vec{\mathbf{a}}\,|\,|\,\vec{\mathbf{b}}. \qquad \textbf{(2.40)}$$

In particular, $\vec{\mathbf{a}}\times\vec{\mathbf{a}}=\vec{\mathbf{0}}$ and then

$$\vec{\mathbf{i}}\times\vec{\mathbf{i}}=\vec{\mathbf{j}}\times\vec{\mathbf{j}}=\vec{\mathbf{k}}\times\vec{\mathbf{k}}=\vec{\mathbf{0}}. \qquad \textbf{(2.41)}$$

Example 2.8. Simplify the expression $(3\vec{\mathbf{i}}\times\vec{\mathbf{j}}-2\vec{\mathbf{k}}\times\vec{\mathbf{i}}+\vec{\mathbf{k}}\times\vec{\mathbf{j}})\times(5\vec{\mathbf{i}}+2\vec{\mathbf{j}})$.

Solution. By (2.37) – (2.39) and (2.41),

$$(3\vec{\mathbf{k}}-2\vec{\mathbf{j}}-\vec{\mathbf{i}})\times(5\vec{\mathbf{i}}+2\vec{\mathbf{j}})=15\vec{\mathbf{k}}\times\vec{\mathbf{i}}+6\vec{\mathbf{k}}\times\vec{\mathbf{j}}-10\vec{\mathbf{j}}\times\vec{\mathbf{i}}-4\vec{\mathbf{j}}\times\vec{\mathbf{j}}-5\vec{\mathbf{i}}\times\vec{\mathbf{i}}-2\vec{\mathbf{i}}\times\vec{\mathbf{j}}=$$

$$=15\vec{\mathbf{j}}-6\vec{\mathbf{i}}+10\vec{\mathbf{k}}-2\vec{\mathbf{k}}=-6\vec{\mathbf{i}}+15\vec{\mathbf{j}}+8\vec{\mathbf{k}}.$$

2.13. One physical application of a vector product: torque

The vector product of two vectors is orthogonal to both of them. Its main use lies in this fact.

So, geometrically, the vector product is useful as a method for constructing a vector perpendicular to a plane if we have two vectors in the plane.

Physically, it appears in the calculation of *torque* and in the calculation of the *magnetic force* on a moving charge. Here we'll consider first of them.

In physics, torque can informally be thought of as "rotational force" or "angular force" which causes a change in rotation motion. This force is defined by linear force multiplied by a radius. Thus, the torque is the rotation analogue of force. The force applied to a lever, multiplied by its distance from the lever's fulcrum, is the torque.

Mathematically, the torque $\vec{\tau}$ is defined as the vector product:

$$\vec{\tau} = \vec{\mathbf{r}} \times \vec{\mathbf{F}}, \tag{2.42}$$

where $\vec{\mathbf{r}}$ is the radius vector from axis of rotation to point of application of the force $\vec{\mathbf{F}}$ as in Fig. 2.24.

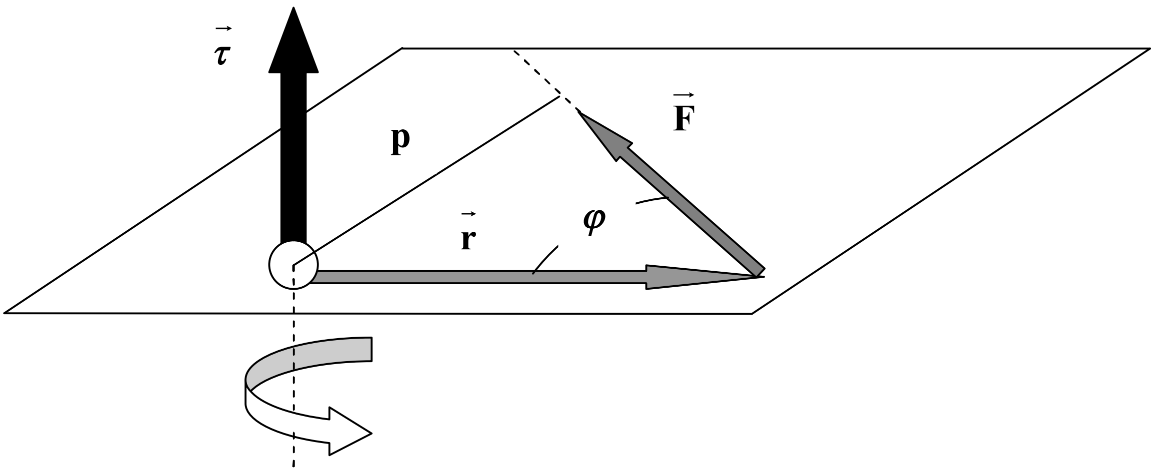


Fig. 2.24

Torque has dimensions of force times distance and the SI units of torque are stated as "Newton meter" $[\mathrm{N} \cdot \mathrm{m}]$ and its direction is determined by the right-handed rule.

A practical way to calculate the magnitude of the torque is to first determine the *lever arm* and then multiply it times the applied force. The lever arm **p** is defined as the perpendicular distance from the axis of rotation to the line of action of the force, i.e. $\mathbf{p} = \left|\vec{\mathbf{r}}\right| \sin\varphi$ [see Fig. 2.24]. Thus,

$$\left|\vec{\tau}\right| = \mathbf{p} \cdot \left|\vec{\mathbf{F}}\right| = \left|\vec{\mathbf{r}}\right|\left|\vec{\mathbf{F}}\right| \sin\varphi . \qquad \textbf{(2.43)}$$

Note that the torque is maximum when the force is perpendicular to the vector $\vec{\mathbf{r}}$, i.e. when $\varphi = \pi/2$.

2.14. Expressing of the vector product in terms of rectangular coordinates

When $\vec{\mathbf{a}} = \{X_a, Y_a, Z_a\}$ and $\vec{\mathbf{b}} = \{X_b, Y_b, Z_b\}$, then we have, by the distributive law and with the account of (2.37) and (2.41),

$$\vec{\mathbf{a}} \times \vec{\mathbf{b}} = (X_a\,\vec{\mathbf{i}} + Y_a\,\vec{\mathbf{j}} + Z_a\,\vec{\mathbf{k}}) \times (X_b\,\vec{\mathbf{i}} + Y_b\,\vec{\mathbf{j}} + Z_b\,\vec{\mathbf{k}}) =$$

$$= X_a X_b\,\vec{\mathbf{i}}\times\vec{\mathbf{i}} + X_a Y_b\,\vec{\mathbf{i}}\times\vec{\mathbf{j}} + X_a Z_b\,\vec{\mathbf{i}}\times\vec{\mathbf{k}} + Y_a X_b\,\vec{\mathbf{j}}\times\vec{\mathbf{i}} + Y_a Y_b\,\vec{\mathbf{j}}\times\vec{\mathbf{j}} +$$

$$+ Y_a Z_b\,\vec{\mathbf{j}}\times\vec{\mathbf{k}} + Z_a X_b\,\vec{\mathbf{k}}\times\vec{\mathbf{i}} + Z_a Y_b\,\vec{\mathbf{k}}\times\vec{\mathbf{j}} + Z_a Z_b\,\vec{\mathbf{k}}\times\vec{\mathbf{k}} =$$

$$= X_a Y_b\,\vec{\mathbf{k}} - X_a Z_b\,\vec{\mathbf{j}} - Y_a X_b\,\vec{\mathbf{k}} + Y_a Z_b\,\vec{\mathbf{i}} + Z_a X_b\,\vec{\mathbf{j}} - Z_a Y_b\,\vec{\mathbf{i}} =$$

$$= (Y_a Z_b - Z_a Y_b)\,\vec{\mathbf{i}} - (X_a Z_b - Z_a X_b)\,\vec{\mathbf{j}} + (X_a Y_b - Y_a X_b)\,\vec{\mathbf{k}} .$$

This result may be written more compactly in the form of determinant:

$$\vec{\mathbf{a}} \times \vec{\mathbf{b}} = \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ X_a & Y_a & Z_a \\ X_b & Y_b & Z_b \end{vmatrix} . \qquad \textbf{(2.44)}$$

Remark. The determinant in (2.44) is usually write down in the form of first-row Laplace expansion (1.13).

Thus the area of parallelogram with sides $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ can be calculated as

$$\mathbf{S} = \left|\vec{\mathbf{a}} \times \vec{\mathbf{b}}\right| = \left\| \begin{matrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ X_a & Y_a & Z_a \\ X_b & Y_b & Z_b \end{matrix} \right\| =$$

$$= \sqrt{(Y_a Z_b - Z_a Y_b)^2 + (X_a Z_b - Z_a X_b)^2 + (X_a Y_b - Y_a X_b)^2} \qquad \textbf{(2.45)}$$

and the area of the triangle with vertices in the points $A\ (x_A,\ y_A,\ z_A)$, $B\ (x_B,\ y_B,\ z_B)$ and $C\ (x_C,\ y_C,\ z_C)$ is given by formula

$$\mathbf{S} = \frac{1}{2}\left|\overrightarrow{\mathbf{AB}} \times \overrightarrow{\mathbf{AC}}\right| = \frac{1}{2}\left|\begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ x_B - x_A & y_B - y_A & z_B - z_A \\ x_C - x_A & y_C - y_A & z_C - z_A \end{vmatrix}\right|. \qquad \mathbf{(2.46)}$$

The condition (2.40) of parallelism of two nonzero vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$, according to 5-th property of determinants [see 2.2 of section 1], now takes a form

$$\frac{X_a}{X_b} = \frac{Y_a}{Y_b} = \frac{Z_a}{Z_b} \Leftrightarrow \vec{\mathbf{a}} \,||\, \vec{\mathbf{b}}. \qquad \mathbf{(2.47)}$$

<u>Example 2.9.</u> Find the magnitude of the torque produced by the force $\vec{\mathbf{F}} = 3\vec{\mathbf{i}} + 4\vec{\mathbf{j}} - 2\vec{\mathbf{k}}$, which applied to the point $M(\,2,\ -3\,,\ 7\,)$, if the initial point of the radius vector is $A(\,2,\ -11,\ 1)$. Find also the lever arm and $\mathbf{sin}\,\boldsymbol{\varphi}$ [Fig. 2.24].

Solution. The radius vector is $\vec{\mathbf{r}} = \overrightarrow{\mathbf{AM}} = \{2-2, -3+11, 7-1\} = \{0, 8, 6\}$. Then by (2.42) and (2.45) we obtain

$$\left|\vec{\boldsymbol{\tau}}\right| = \left|\vec{\mathbf{r}} \times \vec{\mathbf{F}}\right| = \left|\begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ 0 & 8 & 6 \\ 3 & 4 & -2 \end{vmatrix}\right| = \left|-40\vec{\mathbf{i}} + 18\vec{\mathbf{j}} - 24\vec{\mathbf{k}}\right| = \sqrt{40^2 + 18^2 + 24^2} =$$

$= \sqrt{2500} = 50$. The magnitude of the force is $\left|\vec{\mathbf{F}}\right| = \sqrt{3^2 + 4^2 + 2^2} = \sqrt{29}$,

$\left|\vec{\mathbf{r}}\right| = \sqrt{8^2 + 6^2} = 10$ and, by (2.43), the lever arm is $\mathbf{p} = \dfrac{\left|\vec{\boldsymbol{\tau}}\right|}{\left|\vec{\mathbf{F}}\right|} = \dfrac{50}{\sqrt{29}} \approx 9{,}28$ and

the angle which radius vector $\vec{\mathbf{r}}$ makes with respect to the line of action of the force is

$$\sin\varphi = \frac{|\vec{\tau}|}{|\vec{\mathbf{r}}||\vec{\mathbf{F}}|} = \frac{50}{10\sqrt{29}} \approx 0{,}928.$$

Example 2.10. Find the height *CD* of the triangle *ABC* if its sides are $\overrightarrow{\mathbf{AB}} = 3\vec{\mathbf{p}} - 4\vec{\mathbf{q}}$ and $\overrightarrow{\mathbf{BC}} = \vec{\mathbf{p}} + 5\vec{\mathbf{q}}$, where $\vec{\mathbf{p}}$ and $\vec{\mathbf{q}}$ are unit mutually orthogonal vectors.

Solution. The area of the triangle, by (2.36), (2.38), (2.41) and (2.46), is

$$\mathbf{S}_{\Delta\,\mathbf{ABC}} = \frac{1}{2}\left|\overrightarrow{\mathbf{AB}} \times \overrightarrow{\mathbf{BC}}\right| = \frac{1}{2}\left|(3\vec{\mathbf{p}} - 4\vec{\mathbf{q}}) \times (\vec{\mathbf{p}} + 5\vec{\mathbf{q}})\right| = \frac{1}{2}\left|3\vec{\mathbf{p}} \times \vec{\mathbf{p}} + 15\vec{\mathbf{p}} \times \vec{\mathbf{q}} - \right.$$

$$\left. - 4\vec{\mathbf{q}} \times \vec{\mathbf{p}} - 20\vec{\mathbf{q}} \times \vec{\mathbf{q}}\right| = \frac{1}{2}\left|19\,\vec{\mathbf{p}} \times \vec{\mathbf{q}}\right| = \frac{19}{2}\sin\frac{\pi}{2} = 9{,}5.$$ Since $\mathbf{S}_{\Delta\,\mathbf{ABC}} = \dfrac{AB \cdot CD}{2}$ and $AB = \left|\overrightarrow{\mathbf{AB}}\right| = \sqrt{\overrightarrow{\mathbf{AB}} \cdot \overrightarrow{\mathbf{AB}}} = \sqrt{(3\vec{\mathbf{p}} - 4\vec{\mathbf{q}}) \cdot (3\vec{\mathbf{p}} - 4\vec{\mathbf{q}})} = \sqrt{3^2 + 4^2} = 5$ by properties of a scalar product, then $CD = \dfrac{2\mathbf{S}_{\Delta\,\mathbf{ABC}}}{\left|\overrightarrow{\mathbf{AB}}\right|} = \dfrac{19}{5} = 3{,}8.$

2.15. Addition: Triple vector product

Vector multiplication of *three* vectors $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$ and $\vec{\mathbf{c}}$ produce product of the form $\vec{\mathbf{a}} \times (\vec{\mathbf{b}} \times \vec{\mathbf{c}})$ (or $(\vec{\mathbf{a}} \times \vec{\mathbf{b}}) \times \vec{\mathbf{c}}$) which is called ***triple vector product***.

Using (2.44), it is possible to show that

$$\vec{\mathbf{a}} \times (\vec{\mathbf{b}} \times \vec{\mathbf{c}}) = (\vec{\mathbf{a}} \cdot \vec{\mathbf{c}})\vec{\mathbf{b}} - (\vec{\mathbf{a}} \cdot \vec{\mathbf{b}})\vec{\mathbf{c}}. \qquad \textbf{(2.48)}$$

Similarly,
$$(\vec{\mathbf{a}} \times \vec{\mathbf{b}}) \times \vec{\mathbf{c}} = (\vec{\mathbf{a}} \cdot \vec{\mathbf{c}})\vec{\mathbf{b}} - (\vec{\mathbf{b}} \cdot \vec{\mathbf{c}})\vec{\mathbf{a}}. \qquad \textbf{(2.49)}$$

Thus, except when $\vec{\mathbf{b}}$ is perpendicular to both $\vec{\mathbf{a}}$ and $\vec{\mathbf{c}}$,

$$\vec{\mathbf{a}} \times (\vec{\mathbf{b}} \times \vec{\mathbf{c}}) \neq (\vec{\mathbf{a}} \times \vec{\mathbf{b}}) \times \vec{\mathbf{c}},$$

i.e. triple vector product is not associative, and the use of parentheses is necessary.

The following identities are valid:

Jacobi's identity

$$\vec{\mathbf{a}}\times(\vec{\mathbf{b}}\times\vec{\mathbf{c}})+\vec{\mathbf{b}}\times(\vec{\mathbf{c}}\times\vec{\mathbf{a}})+\vec{\mathbf{c}}\times(\vec{\mathbf{a}}\times\vec{\mathbf{b}})=\vec{\mathbf{0}}, \tag{2.50}$$

Lagrange's identity

$$(\vec{\mathbf{a}}\times\vec{\mathbf{b}})\cdot(\vec{\mathbf{c}}\times\vec{\mathbf{d}})=\begin{vmatrix}\vec{\mathbf{a}}\cdot\vec{\mathbf{c}} & \vec{\mathbf{a}}\cdot\vec{\mathbf{d}}\\ \vec{\mathbf{b}}\cdot\vec{\mathbf{c}} & \vec{\mathbf{b}}\cdot\vec{\mathbf{d}}\end{vmatrix}=(\vec{\mathbf{a}}\cdot\vec{\mathbf{c}})(\vec{\mathbf{b}}\cdot\vec{\mathbf{d}})-(\vec{\mathbf{a}}\cdot\vec{\mathbf{d}})(\vec{\mathbf{b}}\cdot\vec{\mathbf{c}}). \tag{2.51}$$

2.16. Concept of a triple scalar product and its geometric meaning

Scalar and vector multiplication of three vectors $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$ and $\vec{\mathbf{c}}$ produce product of the form $(\vec{\mathbf{a}}\times\vec{\mathbf{b}})\cdot\vec{\mathbf{c}}$ (or $\vec{\mathbf{a}}\cdot(\vec{\mathbf{b}}\times\vec{\mathbf{c}})$) which is called ***triple scalar*** (or ***box***, or ***mixed***) ***product***.

Let consider that $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$ and $\vec{\mathbf{c}}$ are three noncoplanar vectors, which form right-handed triad as in Fig. 2.25.

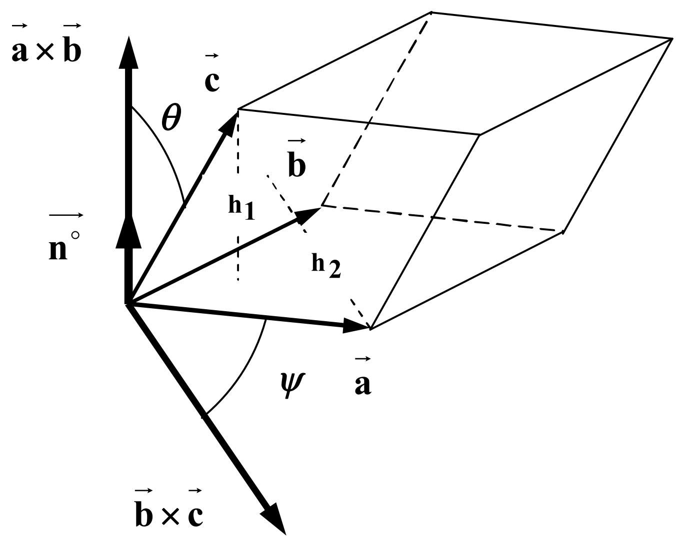


Fig. 2.25

Denote by $\boldsymbol{\theta}$ the smaller angle between $\vec{\mathbf{c}}$ and $\vec{\mathbf{a}}\times\vec{\mathbf{b}}$. Then the triple scalar product $(\vec{\mathbf{a}}\times\vec{\mathbf{b}})\cdot\vec{\mathbf{c}}$ is by definitions (2.15) and (2.34)

$$(\vec{\mathbf{a}}\times\vec{\mathbf{b}})\cdot\vec{\mathbf{c}}=|\vec{\mathbf{a}}||\vec{\mathbf{b}}|\sin(\vec{\mathbf{a}},\hat{\ }\vec{\mathbf{b}})\ \vec{\mathbf{n}}^\circ\cdot\vec{\mathbf{c}}=|\vec{\mathbf{a}}||\vec{\mathbf{b}}|\sin(\vec{\mathbf{a}},\hat{\ }\vec{\mathbf{b}})\,|\vec{\mathbf{c}}|\cos\boldsymbol{\theta}.$$

Since $|\vec{\mathbf{a}}||\vec{\mathbf{b}}|\sin(\vec{\mathbf{a}}\,\hat{,}\,\vec{\mathbf{b}})$ is the area $\mathbf{S}_1$ of parallelogram formed by the vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ and $|\vec{\mathbf{c}}|\cos\theta = \mathbf{Pr}_{\vec{\mathbf{a}}\times\vec{\mathbf{b}}}\vec{\mathbf{c}} = \mathbf{h}_1$, where $0 \le \theta < \pi/2$, is corresponding height of parallelepiped formed by $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$ and $\vec{\mathbf{c}}$, then $(\vec{\mathbf{a}}\times\vec{\mathbf{b}})\cdot\vec{\mathbf{c}} = \mathbf{S}_1\cdot\mathbf{h}_1$ is numerically equal to volume $\mathbf{V}$ of this parallelepiped.

Similarly, if we denote by ψ the smaller angle between $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}\times\vec{\mathbf{c}}$ [see Fig. 2.25], then we'll receive:

$$\vec{\mathbf{a}}\cdot(\vec{\mathbf{b}}\times\vec{\mathbf{c}}) = |\vec{\mathbf{a}}|\cos\psi\,|\vec{\mathbf{b}}||\vec{\mathbf{c}}|\sin(\vec{\mathbf{b}}\,\hat{,}\,\vec{\mathbf{c}})\cdot\vec{\mathbf{c}} = \mathbf{S}_2\cdot\mathbf{h}_2,$$

where $\mathbf{S}_2 = |\vec{\mathbf{b}}||\vec{\mathbf{c}}|\sin(\vec{\mathbf{b}}\,\hat{,}\,\vec{\mathbf{c}})$ is the area of parallelogram formed by the vectors $\vec{\mathbf{b}}$ and $\vec{\mathbf{c}}$ and $\mathbf{h}_2 = |\vec{\mathbf{a}}|\cos\psi = \mathbf{Pr}_{\vec{\mathbf{b}}\times\vec{\mathbf{c}}}\vec{\mathbf{a}}$ $(0 \le \psi < \pi/2)$ is corresponding height of the same parallelepiped.

Thus, $(\vec{\mathbf{a}}\times\vec{\mathbf{b}})\cdot\vec{\mathbf{c}} = \vec{\mathbf{a}}\cdot(\vec{\mathbf{b}}\times\vec{\mathbf{c}})$, i.e. in a scalar triple product the dot and cross, which denote a scalar and vector products respectively, can be interchanged without affecting the result. Since the parenthesis in a triple scalar product are not necessary and usually omitted, the more so, as writing $(\vec{\mathbf{a}}\cdot\vec{\mathbf{b}})\times\vec{\mathbf{c}}$ is without meaning. Therefore triple scalar product can be written in the simplify form $\vec{\mathbf{a}}\vec{\mathbf{b}}\vec{\mathbf{c}}$.

2.17. Geometric properties of a triple scalar product

As it easy to see, if the vectors $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$ and $\vec{\mathbf{c}}$ formed *left-handed* triad as in Fig. 2.26, then $\pi/2 < \theta \le \pi$. Hence, in this case $\mathbf{h}_1 = -\,\mathbf{Pr}_{\vec{\mathbf{a}}\times\vec{\mathbf{b}}}\vec{\mathbf{c}}$, and therefore $\mathbf{V} = -\,(\vec{\mathbf{a}}\times\vec{\mathbf{b}})\cdot\vec{\mathbf{c}}$.

Thus,

∘ the triple scalar product is in absolute value numerically equal to the volume of a parallelepiped formed by $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$ and $\vec{\mathbf{c}}$:

$$\mathbf{V} = \left|\vec{\mathbf{a}}\vec{\mathbf{b}}\vec{\mathbf{c}}\right|; \qquad \textbf{(2.52)}$$

∘ if the vectors $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$ and $\vec{\mathbf{c}}$ formed *right-handed* triad, then $\vec{\mathbf{a}}\vec{\mathbf{b}}\vec{\mathbf{c}} > 0$, and conversely;

∘ if the vectors $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$ and $\vec{\mathbf{c}}$ formed *left-handed* triad, then $\vec{\mathbf{a}}\vec{\mathbf{b}}\vec{\mathbf{c}} < 0$, and conversely.

Note. The volume of the pyramid $\mathbf{V_{pyr}}$ formed by the vectors $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$ and $\vec{\mathbf{c}}$ numerically is equal to **V**/ 6.

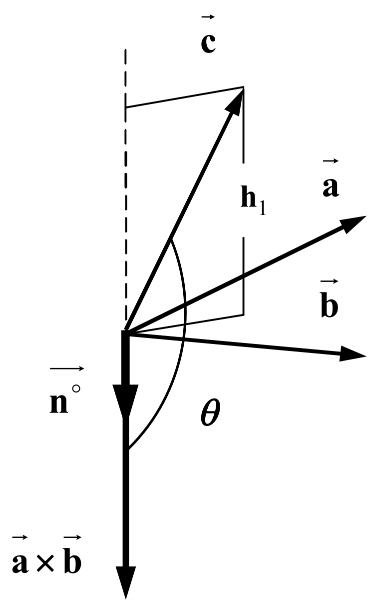


Fig. 2.26

It is obvious that if the vectors $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$ and $\vec{\mathbf{c}}$ are noncoplanar then $\vec{\mathbf{a}}\vec{\mathbf{b}}\vec{\mathbf{c}} \neq 0$ (and conversely), and if the vectors $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$ and $\vec{\mathbf{c}}$ are coplanar then **V** = 0, i.e. $\vec{\mathbf{a}}\vec{\mathbf{b}}\vec{\mathbf{c}} = 0$, and conversely, i.e.

$$(\vec{\mathbf{a}}, \vec{\mathbf{b}}, \vec{\mathbf{c}} \text{ are coplanar}) \Leftrightarrow \vec{\mathbf{a}}\vec{\mathbf{b}}\vec{\mathbf{c}} = 0. \qquad \textbf{(2.53)}$$

Remark. In particular, *the triple scalar product is equal to zero if only one factor is zero vector or if only two factors are parallel (or equal each other) vectors.*

2.18. Expressing of the triple scalar product in terms of rectangular coordinates

Let $\vec{\mathbf{a}} = \{X_a, Y_a, Z_a\}$, $\vec{\mathbf{b}} = \{X_b, Y_b, Z_b\}$ and $\vec{\mathbf{c}} = \{X_c, Y_c, Z_c\}$. Then, by (2.44),

$$\vec{\mathbf{a}}\cdot(\vec{\mathbf{b}}\times\vec{\mathbf{c}}) = (X_a\,\vec{\mathbf{i}} + Y_a\,\vec{\mathbf{j}} + Z_a\,\vec{\mathbf{k}})\begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ X_b & Y_b & Z_b \\ X_c & Y_c & Z_c \end{vmatrix} = (X_a\,\vec{\mathbf{i}} + Y_a\,\vec{\mathbf{j}} + Z_a\,\vec{\mathbf{k}})\,\cdot$$

$$\cdot\,[(Y_b Z_c - Z_b Y_c)\,\vec{\mathbf{i}} - (X_b Z_c - Z_b X_c)\,\vec{\mathbf{j}} + (X_b Y_c - Y_b X_c)\,\vec{\mathbf{k}}] =$$

$$= X_a(Y_b Z_c - Z_b Y_c) - Y_a(X_b Z_c - Z_b X_c) + Z_a(X_b Y_c - Y_b X_c).$$

This result may be written more compactly in the form of a determinant:

$$\vec{\mathbf{a}}\vec{\mathbf{b}}\vec{\mathbf{c}} = \begin{vmatrix} X_a & Y_a & Z_a \\ X_b & Y_b & Z_b \\ X_c & Y_c & Z_c \end{vmatrix}. \qquad \textbf{(2.54)}$$

2.19. Algebraic properties of a triple scalar product

All properties of the triple scalar product, including its geometrical properties, follow from the properties of the determinants [see 2.2 of section 1]. For example,

$$\vec{\mathbf{a}}\vec{\mathbf{b}}\vec{\mathbf{c}} = \begin{vmatrix} X_a & Y_a & Z_a \\ X_b & Y_b & Z_b \\ X_c & Y_c & Z_c \end{vmatrix} = -\begin{vmatrix} X_b & Y_b & Z_b \\ X_a & Y_a & Z_a \\ X_c & Y_c & Z_c \end{vmatrix} = -\vec{\mathbf{b}}\vec{\mathbf{a}}\vec{\mathbf{c}} = \begin{vmatrix} X_b & Y_b & Z_b \\ X_c & Y_c & Z_c \\ X_a & Y_a & Z_a \end{vmatrix} = \vec{\mathbf{b}}\vec{\mathbf{c}}\vec{\mathbf{a}}.$$

Similarly, we can prove that $\vec{\mathbf{a}}\vec{\mathbf{b}}\vec{\mathbf{c}} = \vec{\mathbf{c}}\vec{\mathbf{a}}\vec{\mathbf{b}} = -\vec{\mathbf{a}}\vec{\mathbf{c}}\vec{\mathbf{b}} = -\vec{\mathbf{c}}\vec{\mathbf{b}}\vec{\mathbf{a}}$. Thus,

$$\vec{\mathbf{a}}\vec{\mathbf{b}}\vec{\mathbf{c}} = \vec{\mathbf{b}}\vec{\mathbf{c}}\vec{\mathbf{a}} = \vec{\mathbf{c}}\vec{\mathbf{a}}\vec{\mathbf{b}} \qquad \textbf{(2.55)}$$

and

$$\vec{\mathbf{a}}\vec{\mathbf{b}}\vec{\mathbf{c}} = -\vec{\mathbf{b}}\vec{\mathbf{a}}\vec{\mathbf{c}} = -\vec{\mathbf{a}}\vec{\mathbf{c}}\vec{\mathbf{b}} = -\vec{\mathbf{c}}\vec{\mathbf{b}}\vec{\mathbf{a}}, \qquad \textbf{(2.56)}$$

i.e. the triple scalar product does not change at cyclic (circular) rearrangement of the vectors, but it changes the sign at the rearrangement of any two neighboring vectors.

The following laws are valid:

Associative Law for scalar factor

$$[(\lambda\vec{\mathbf{a}})\times\vec{\mathbf{b}}]\cdot\vec{\mathbf{c}} = [\vec{\mathbf{a}}\times(\lambda\vec{\mathbf{b}})]\cdot\vec{\mathbf{c}} = (\vec{\mathbf{a}}\times\vec{\mathbf{b}})\cdot(\lambda\vec{\mathbf{c}}) = \lambda(\vec{\mathbf{a}}\vec{\mathbf{b}}\vec{\mathbf{c}}), \qquad \textbf{(2.57)}$$

Distributive Law

$$(\vec{\mathbf{a}}\times\vec{\mathbf{b}})\cdot(\vec{\mathbf{c}}+\vec{\mathbf{d}}) = \vec{\mathbf{a}}\vec{\mathbf{b}}\vec{\mathbf{c}} + \vec{\mathbf{a}}\vec{\mathbf{b}}\vec{\mathbf{d}}, \qquad \textbf{(2.58)}$$

$$(\vec{\mathbf{a}}+\vec{\mathbf{b}})\cdot(\vec{\mathbf{c}}\times\vec{\mathbf{d}}) = \vec{\mathbf{a}}\vec{\mathbf{c}}\vec{\mathbf{d}} + \vec{\mathbf{b}}\vec{\mathbf{c}}\vec{\mathbf{d}}, \qquad \textbf{(2.59)}$$

$$(\lambda+\mu)(\vec{\mathbf{a}}\vec{\mathbf{b}}\vec{\mathbf{c}}) = \lambda\,\vec{\mathbf{a}}\vec{\mathbf{b}}\vec{\mathbf{c}} + \mu\vec{\mathbf{a}}\vec{\mathbf{b}}\vec{\mathbf{c}}. \qquad \textbf{(2.60)}$$

Example 2.11. Simplify the expression $(\vec{\mathbf{a}}+\vec{\mathbf{b}})\cdot[(\vec{\mathbf{b}}+\vec{\mathbf{c}})\times(\vec{\mathbf{c}}+\vec{\mathbf{a}})]$.

Solution. Using the properties of the vector product and the triple scalar product and, in particular, taking into account the remark to (2.53), we obtain

$$(\vec{\mathbf{a}}+\vec{\mathbf{b}})\cdot(\vec{\mathbf{b}}\times\vec{\mathbf{c}}+\vec{\mathbf{b}}\times\vec{\mathbf{a}}+\vec{\mathbf{c}}\times\vec{\mathbf{c}}+\vec{\mathbf{c}}\times\vec{\mathbf{a}}) = (\vec{\mathbf{a}}+\vec{\mathbf{b}})\cdot(\vec{\mathbf{b}}\times\vec{\mathbf{c}}+\vec{\mathbf{b}}\times\vec{\mathbf{a}}+\vec{\mathbf{c}}\times\vec{\mathbf{a}}) =$$

$$= \vec{\mathbf{a}}\vec{\mathbf{b}}\vec{\mathbf{c}} + \vec{\mathbf{b}}\vec{\mathbf{c}}\vec{\mathbf{a}} = 2\vec{\mathbf{a}}\vec{\mathbf{b}}\vec{\mathbf{c}}.$$

2.20. Some applications of a triple scalar product

Let $\vec{\mathbf{a}} = \{X_a, Y_a, Z_a\}$, $\vec{\mathbf{b}} = \{X_b, Y_b, Z_b\}$ and $\vec{\mathbf{c}} = \{X_c, Y_c, Z_c\}$. Then, as follows from the geometric properties of a triple scalar product and from (2.54), the volume of a parallelepiped can be calculated by the formula

$$\mathbf{V} = \left|\begin{vmatrix} X_a & Y_a & Z_a \\ X_b & Y_b & Z_b \\ X_c & Y_c & Z_c \end{vmatrix}\right|, \qquad \textbf{(2.61)}$$

and also $(\vec{\mathbf{a}}, \vec{\mathbf{b}}, \vec{\mathbf{c}}$ are noncoplanar) $\Leftrightarrow \begin{vmatrix} X_a & Y_a & Z_a \\ X_b & Y_b & Z_b \\ X_c & Y_c & Z_c \end{vmatrix} \neq 0$, in particular,

$$(\vec{\mathbf{a}},\vec{\mathbf{b}},\vec{\mathbf{c}} \text{ formed right-handed system}) \Leftrightarrow \begin{vmatrix} X_a & Y_a & Z_a \\ X_b & Y_b & Z_b \\ X_c & Y_c & Z_c \end{vmatrix} > 0, \qquad \textbf{(2.62)}$$

$$(\vec{\mathbf{a}},\vec{\mathbf{b}},\vec{\mathbf{c}} \text{ formed left-handed system}) \Leftrightarrow \begin{vmatrix} X_a & Y_a & Z_a \\ X_b & Y_b & Z_b \\ X_c & Y_c & Z_c \end{vmatrix} < 0. \qquad \textbf{(2.63)}$$

The condition of coplanarity of three vectors (2.53) now takes a form

$$(\vec{\mathbf{a}},\vec{\mathbf{b}},\vec{\mathbf{c}} \text{ are coplanar}) \Leftrightarrow \begin{vmatrix} X_a & Y_a & Z_a \\ X_b & Y_b & Z_b \\ X_c & Y_c & Z_c \end{vmatrix} = 0. \qquad \textbf{(2.64)}$$

The volume of the pyramid with vertices in the points A (x_A, y_A, z_A), B (x_B, y_B, z_B), C (x_C, y_C, z_C) and D (x_D, y_D, z_D) is given by the formula

$$\mathbf{V_{pyr}} = \frac{1}{6}\left|\overrightarrow{\mathbf{AB}}\,\overrightarrow{\mathbf{AC}}\,\overrightarrow{\mathbf{AD}}\right| = \frac{1}{6}\left|\begin{vmatrix} x_B - x_A & y_B - y_A & z_B - z_A \\ x_C - x_A & y_C - y_A & z_C - z_A \\ x_D - x_A & y_D - y_A & z_D - z_A \end{vmatrix}\right| =$$

$$= \frac{1}{6}\left|\begin{vmatrix} x_A & y_A & z_A & 1 \\ x_B & y_B & z_B & 1 \\ x_C & y_C & z_C & 1 \\ x_D & y_D & z_D & 1 \end{vmatrix}\right|. \qquad \textbf{(2.65)}$$

Note. If the points A, B, C and D all lie in the same plane, then

$$\begin{vmatrix} x_A & y_A & z_A & 1 \\ x_B & y_B & z_B & 1 \\ x_C & y_C & z_C & 1 \\ x_D & y_D & z_D & 1 \end{vmatrix} = 0, \text{ and conversely.}$$

The following identities are valid:

$$\textbf{1.}\ (\vec{\mathbf{a}}\times\vec{\mathbf{b}})\times(\vec{\mathbf{c}}\times\vec{\mathbf{d}}) = \begin{vmatrix} \vec{\mathbf{a}}\vec{\mathbf{c}}\vec{\mathbf{d}} & \vec{\mathbf{b}}\vec{\mathbf{c}}\vec{\mathbf{d}} \\ \vec{\mathbf{a}} & \vec{\mathbf{b}} \end{vmatrix} = \vec{\mathbf{b}}(\vec{\mathbf{a}}\vec{\mathbf{c}}\vec{\mathbf{d}}) - \vec{\mathbf{a}}(\vec{\mathbf{b}}\vec{\mathbf{c}}\vec{\mathbf{d}}). \qquad \textbf{(2.66)}$$

$$2.\ (\vec{\mathbf{a}}\vec{\mathbf{b}}\vec{\mathbf{c}})\cdot(\vec{\mathbf{d}}\vec{\mathbf{e}}\vec{\mathbf{f}}) = \begin{vmatrix} \vec{\mathbf{a}}\cdot\vec{\mathbf{d}} & \vec{\mathbf{a}}\cdot\vec{\mathbf{e}} & \vec{\mathbf{a}}\cdot\vec{\mathbf{f}} \\ \vec{\mathbf{b}}\cdot\vec{\mathbf{d}} & \vec{\mathbf{b}}\cdot\vec{\mathbf{e}} & \vec{\mathbf{b}}\cdot\vec{\mathbf{f}} \\ \vec{\mathbf{c}}\cdot\vec{\mathbf{d}} & \vec{\mathbf{c}}\cdot\vec{\mathbf{e}} & \vec{\mathbf{c}}\cdot\vec{\mathbf{f}} \end{vmatrix}. \qquad \mathbf{(2.67)}$$

In particular, with taking into account the scalar product's commutativity, we obtain

$$(\vec{\mathbf{a}}\vec{\mathbf{b}}\vec{\mathbf{c}})^2 = (\vec{\mathbf{a}}\vec{\mathbf{b}}\vec{\mathbf{c}})\cdot(\vec{\mathbf{a}}\vec{\mathbf{b}}\vec{\mathbf{c}}) = \begin{vmatrix} |\vec{\mathbf{a}}|^2 & \vec{\mathbf{a}}\cdot\vec{\mathbf{b}} & \vec{\mathbf{a}}\cdot\vec{\mathbf{c}} \\ \vec{\mathbf{a}}\cdot\vec{\mathbf{b}} & |\vec{\mathbf{b}}|^2 & \vec{\mathbf{b}}\cdot\vec{\mathbf{c}} \\ \vec{\mathbf{a}}\cdot\vec{\mathbf{c}} & \vec{\mathbf{b}}\cdot\vec{\mathbf{c}} & |\vec{\mathbf{c}}|^2 \end{vmatrix} =$$

$$= \left(|\vec{\mathbf{a}}||\vec{\mathbf{b}}||\vec{\mathbf{c}}|\right)^2 + 2(\vec{\mathbf{a}}\cdot\vec{\mathbf{b}})(\vec{\mathbf{b}}\cdot\vec{\mathbf{c}})(\vec{\mathbf{a}}\cdot\vec{\mathbf{c}}) - |\vec{\mathbf{a}}|^2(\vec{\mathbf{b}}\cdot\vec{\mathbf{c}})^2 - |\vec{\mathbf{b}}|^2(\vec{\mathbf{a}}\cdot\vec{\mathbf{c}})^2 -$$

$$- |\vec{\mathbf{c}}|^2(\vec{\mathbf{a}}\cdot\vec{\mathbf{b}})^2.$$

The last ratio can be applied to the calculation of the volume of the parallelepiped with the edges *a, b, c*, which come out from the same vertex, and planar angles between them $\boldsymbol{\alpha} = (a \hat{,} b)$, $\boldsymbol{\beta} = (b \hat{,} c)$, $\boldsymbol{\gamma} = (a \hat{,} c)$ as it shown in Fig. 2.27.

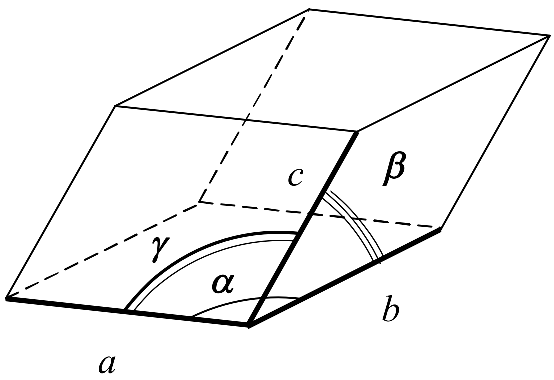


Fig. 2.27

Then

$$\mathbf{V} = abc\sqrt{\left|\ 1 + 2\cos\alpha\cos\beta\cos\gamma - \cos^2\alpha - \cos^2\beta - \cos^2\gamma\ \right|}. \qquad \mathbf{(2.68)}$$

Example 2.12. Show that the vectors $\vec{\mathbf{a}} = -4\vec{\mathbf{i}} - 3\vec{\mathbf{j}} - 9\vec{\mathbf{k}}$, $\vec{\mathbf{b}} = \vec{\mathbf{i}} - \vec{\mathbf{k}}$, $\vec{\mathbf{c}} = -5\vec{\mathbf{i}} - 4\vec{\mathbf{j}} + 3\vec{\mathbf{k}}$ are noncoplanar and determine the orientation of respective system, formed by these vectors.

Solution. By (2.54), $\vec{\mathbf{a}}\vec{\mathbf{b}}\vec{\mathbf{c}} = \begin{vmatrix} -4 & -3 & -9 \\ 1 & 0 & -1 \\ -5 & -4 & 3 \end{vmatrix} = -15 + 36 + 9 + 16 = 46$.

Since $\vec{\mathbf{a}}\vec{\mathbf{b}}\vec{\mathbf{c}} \neq 0$ then the vectors $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$ and $\vec{\mathbf{c}}$ are noncoplanar. Since also $\vec{\mathbf{a}}\vec{\mathbf{b}}\vec{\mathbf{c}} > 0$ then, according to (2.62), the vectors $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$ and $\vec{\mathbf{c}}$ formed right-handed system.

Example 2.13. Find the height *DE* of the pyramid with vertices in the points *A* (2, 3, 1), *B* (4, 1, -2), *C* (6, 3, 7) and *D* (-5, -4, 8).

Solution. By (2.65),

$$\mathbf{V_{pyr}} = \frac{1}{6}\left|\overrightarrow{\mathbf{AB}}\,\overrightarrow{\mathbf{AC}}\,\overrightarrow{\mathbf{AD}}\right| = \frac{1}{6}\left|\begin{vmatrix} 2 & -2 & -3 \\ 4 & 0 & 6 \\ -7 & -7 & 7 \end{vmatrix}\right| = \frac{1}{6}\left|84 + 84 + 56 + 84\right| = \frac{154}{3}.$$

Also $\mathbf{V_{pyr}} = \frac{1}{3}\mathbf{S}_{\Delta\,\mathbf{ABC}} \cdot DE$, from which $DE = \dfrac{3\mathbf{V_{pyr}}}{\mathbf{S}_{\Delta\,\mathbf{ABC}}}$. For the calculating the area of a triangle *ABC* we'll use the formula (2.35): $\mathbf{S}_{\Delta\,\mathbf{ABC}} = \frac{1}{2}\left|\overrightarrow{\mathbf{AB}} \times \overrightarrow{\mathbf{AC}}\right| =$

$$= \frac{1}{2}\sqrt{\left|\overrightarrow{\mathbf{AB}}\right|^2\left|\overrightarrow{\mathbf{AC}}\right|^2 - (\overrightarrow{\mathbf{AB}} \cdot \overrightarrow{\mathbf{AC}})^2} = \frac{1}{2}\sqrt{(4+4+9)\cdot(16+36) - (8-18)^2} =$$

$$= \frac{1}{2}\sqrt{17 \cdot 52 - 100} = 14.$$ Thus, $DE = \dfrac{154}{14} = 11$.

Example 2.14. Find the volume of the parallelepiped with the edges $a = 2\sqrt{3}$, $b = 5$, $c = 6$, which come out from the same vertex, and planar angles between them $\boldsymbol{\alpha} = (a\,\hat{,}\,b) = \dfrac{2\pi}{3}$, $\boldsymbol{\beta} = (b\,\hat{,}\,c) = \dfrac{\pi}{4}$, $\boldsymbol{\gamma} = (a\,\hat{,}\,c) = \dfrac{\pi}{4}$.

Solution. By (2.68),

$$\mathbf{V} = 2\sqrt{3}\cdot 5 \cdot 6 \cdot \sqrt{\left|\, 1 + 2\cdot(-\frac{1}{2})\cdot\frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}} - \frac{1}{2} - \frac{1}{4} - \frac{1}{2}\,\right|} = 60\sqrt{3}\cdot\frac{\sqrt{3}}{2} = 90.$$

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Part 1

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